Suggested home exercises – lecture 10

10.1.9 Analyze the map $x_{n+1} = 2x_n/(1 + x_n)$ for both positive and negative x_n.

10.1.10 Show that the map $x_{n+1} = 1 + \frac{1}{2}\sin x_n$ has a unique fixed point. Is it stable?

10.1.11 (Cubic map) Consider the map $x_{n+1} = 3x_n - x_n^3$.

a) Find all the fixed points and classify their stability.

b) Draw a cobweb starting at $x_0 = 1.9$.

c) Draw a cobweb starting at $x_0 = 2.1$.

d) Try to explain the dramatic difference between the orbits found in parts (b) and (c). For instance, can you prove that the orbit in (b) will remain bounded for all n? Or that $|x_n| \to \infty$ in (c)?

10.3.3 Analyze the long-term behavior of the map $x_{n+1} = rx_n/(1 + x_n^2)$, where $r > 0$. Find and classify all fixed points as a function of r. Can there be periodic solutions? Chaos?

10.4.1 (Exponential map) Consider the map $x_{n+1} = r \exp x_n$ for $r > 0$.

a) Analyze the map by drawing a cobweb.

b) Show that a tangent bifurcation occurs at $r = 1/e$.

c) Sketch the time series x_n vs. n for r just above and just below $r = 1/e$.

10.4.5 (Band merging and crisis) Show numerically that the period-doubling bifurcations of the 3-cycle for the logistic map accumulate near $r = 3.8495\ldots$, to form three small chaotic bands. Show that these chaotic bands merge near $r = 3.857\ldots$ to form a much larger attractor that nearly fills an interval.

This discontinuous jump in the size of an attractor is an example of a ***crisis*** (Grebogi, Ott, and Yorke 1983a).

10.5.1 Calculate the Liapunov exponent for the linear map $x_{n+1} = rx_n$.

$x_{n+1} = r \sin \pi x_n$, where $0 < r \leq 1$

10.6.1 (Naive approach)

a) At each of 200 equally spaced r values, plot x_{700} through x_{1000} vertically above r, starting from some random initial condition x_0. Check your orbit diagram against Figure 10.6.2 to be sure your program is working.

b) Now go to finer resolution near the period-doubling bifurcations, and estimate r_n, for $n = 1, 2, \ldots, 6$. Try to achieve five significant figures of accuracy.

c) Use the numbers from (b) to estimate the Feigenbaum ratio $\dfrac{r_n - r_{n-1}}{r_{n+1} - r_n}$.

(Note: To get accurate estimates in part (b), you need to be clever, or careful, or both. As you probably found, a straightforward approach is hampered by "critical slowing down"—the convergence to a cycle becomes unbearably slow when that cycle is on the verge of period-doubling. This makes it hard to decide precisely where the bifurcation occurs. To achieve the desired accuracy, you may have to use double precision arithmetic, and about 10^4 iterates. But maybe you can find a shortcut by reformulating the problem.)

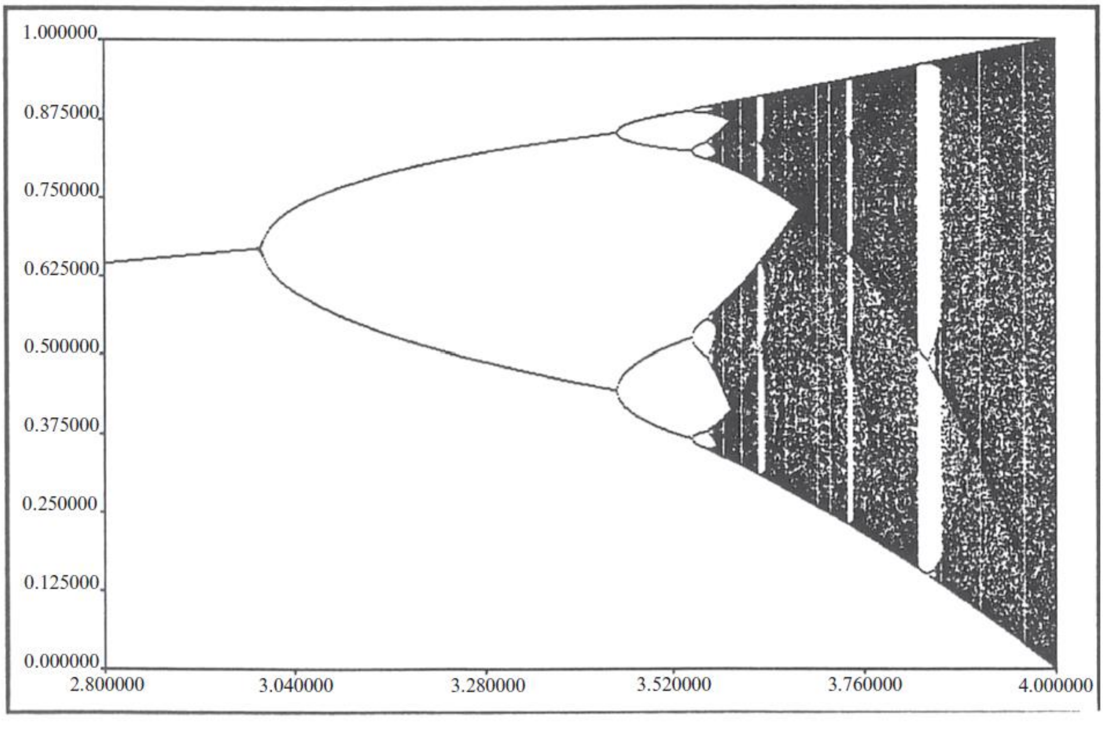

Figure 10.6.2 Courtesy of Andy Christian