Suggested home exercises – lecture 11

11.3.1 (Middle-halves Cantor set) Construct a new kind of Cantor set by removing the middle half of each sub-interval, rather than the middle third.
a) Find the similarity dimension of the set.
b) Find the measure of the set.

11.3.7 (Snowflake) To construct the famous fractal known as the ***von Koch snowflake curve***, use an equilateral triangle for S_0. Then do the von Koch procedure of Figure 11.3.1 on each of the three sides.
a) Show that S_1 looks like a star of David.
b) Draw S_2 and S_3.
c) The snowflake is the limiting curve $S = S_\infty$. Show that it has infinite arc length.
d) Find the area of the region enclosed by S.
e) Find the similarity dimension of S.

The snowflake curve is continuous but nowhere differentiable—loosely speaking, it is "all corners"!

11.4.7 (A lopsided fractal) Divide the closed unit interval [0,1] into four quarters. Delete the open second quarter from the left. This produces a set S_1. Repeat this construction indefinitely; i.e., generate S_{n+1} from S_n by deleting the second quarter of each of the intervals in S_n.
a) Sketch the sets $S_1, \ldots, S_4$.
b) Compute the box dimension of the limiting set S_∞.
c) Is S_∞ self-similar?

12.1.7 (Smale horseshoe) Figure 1 illustrates the mapping known as the ***Smale horseshoe*** (Smale 1967).

12.2.7 (2-cycle) Consider the Hénon map with $-1 < b < 1$. Show that the map has a 2-cycle for $a > a_1 = \frac{3}{4}(1-b)^2$. For which values of a is the 2-cycle stable?

12.1.7 (Smale horseshoe) Figure 1 illustrates the mapping known as the ***Smale horseshoe*** (Smale 1967).

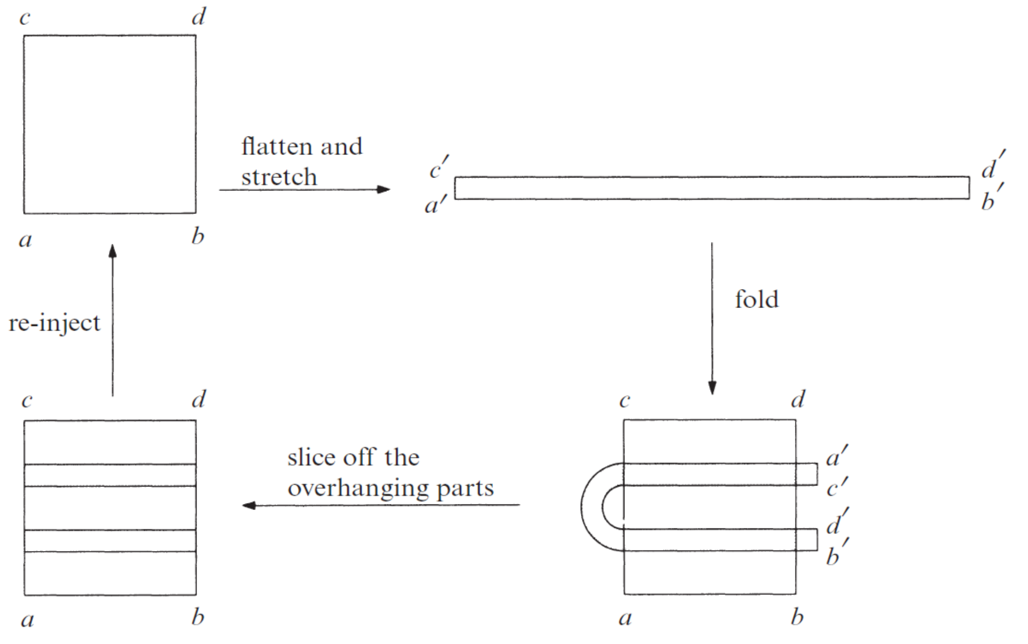

Figure 1

Notice the crucial difference between this map and that shown in Figure 12.1.3: here the horseshoe *hangs over the edge* of the original square. The overhanging parts are lopped off before the next iteration proceeds.

a) The square at the lower left of Figure 1 contains two shaded horizontal strips. Find the points in the original square that map to these strips. (These are the points that survive one iteration, in the sense that they still remain in the square.)
b) Show that after the next round of the mapping, the square on the lower left contains *four* horizontal strips. Find where *they* came from in the original square. (These are the points that survive two iterations.)
c) Describe the set of points in the original square that survive forever.

The horseshoe arises naturally in the analysis of *transient chaos* in differential equations. Roughly speaking, the Poincaré map of such systems can often be approximated by the horseshoe. During the time the orbit remains in a certain region corresponding to the square above, the stretching and folding of the map causes chaos. However, almost all orbits get mapped out of this region eventually (into the "overhang"), and then they escape to some distant part of phase space; this is why the chaos is only *transient*. See Guckenheimer and Holmes (1983) or Arrowsmith and Place (1990) for introductions to the mathematics of horseshoes.