Suggested home exercises – lecture 2

2.2 Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1 $\dot{x} = 4x^2 - 16$

2.4 Linear Stability Analysis

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

2.4.7 $\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

3.1 Saddle-Node Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

3.1.1 $\dot{x} = 1 + rx + x^2$

3.3 Laser Threshold

3.3.1 (An improved model of a laser) In the simple laser model considered in Section 3.3, we wrote an *algebraic* equation relating N, the number of excited atoms, to n, the number of laser photons. In more realistic models, this would be replaced by a *differential* equation. For instance, Milonni and Eberly (1988) show that after certain reasonable approximations, quantum mechanics leads to the system

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p.$$

Here G is the gain coefficient for stimulated emission, k is the decay rate due to loss of photons by mirror transmission, scattering, etc., f is the decay rate for spontaneous emission, and p is the pump strength. All parameters are positive, except p, which can have either sign.

This two-dimensional system will be analyzed in Exercise 8.1.13. For now, let's convert it to a one-dimensional system, as follows.

a) Suppose that N relaxes much more rapidly than n. Then we may make the quasi-static approximation $\dot{N} \approx 0$. Given this approximation, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n. (This procedure is often called ***adiabatic elimination***, and one says that the evolution of $N(t)$ is *slaved* to that of $n(t)$. See Haken (1983).)
b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.
c) What type of bifurcation occurs at the laser threshold p_c?
d) (Hard question) For what range of parameters is it valid to make the approximation used in (a)?