Suggested home exercises – lecture 3

3.5.7 (Nondimensionalizing the logistic equation) Consider the logistic equation $\dot{N} = rN(1 - N/K)$, with initial condition $N(0) = N_0$.

a) This system has three dimensional parameters r, K, and N_0. Find the dimensions of each of these parameters.
b) Show that the system can be rewritten in the dimensionless form

$$\frac{dx}{d\tau} = x(1-x), \quad x(0) = x_0$$

for appropriate choices of the dimensionless variables x, x_0, and τ.
c) Find a different nondimensionalization in terms of variables u and τ, where u is chosen such that the initial condition is always $u_0 = 1$.
d) Can you think of any advantage of one nondimensionalization over the other?

3.7.3 (A model of a fishery) The equation $\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry—they simply catch the same number of fish every day.)
a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1-x) - h,$$

for suitably defined dimensionless quantities x, τ, and h.
b) Plot the vector field for different values of h.
c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.
d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

There's something silly about this model—the population can become negative! A better model would have a fixed point at zero population for all values of H. See the next exercise for such an improvement.

3.7.5 (A biochemical switch) Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. Explaining the development of these patterns is one of the outstanding problems of biology; see Murray (2003) for an excellent review.

As one ingredient in a model of pattern formation, Lewis et al. (1977) considered a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, the gene may normally be inactive but can be "switched on" to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

where the k's are positive constants. The production of g is stimulated by s_0 at a rate k_1, and by an *autocatalytic* or positive feedback process (the nonlinear term). There is also a linear degradation of g at a rate k_2.

a) Show that the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$$

where $r > 0$ and $s \geq 0$ are dimensionless groups.

b) Show that if $s = 0$, there are two positive fixed points x^* if $r < r_c$, where r_c is to be determined.

c) Assume that initially there is no gene product, i.e., $g(0) = 0$, and suppose s is slowly increased from zero (the activating signal is turned on); what happens to $g(t)$? What happens if s then goes back to zero? Does the gene turn off again?

d) Find parametric equations for the bifurcation curves in (r, s) space, and classify the bifurcations that occur.

e) Use the computer to give a quantitatively accurate plot of the stability diagram in (r, s) space.

For further discussion of this model, see Lewis et al. (1977); Edelstein–Keshet (1988), Section 7.5; or Murray (2002), Chapter 6.