Suggested home exercises – lecture 4

5.1.9 Consider the system $\dot{x} = -y$, $\dot{y} = -x$.

a) Sketch the vector field.

b) Show that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$. (Hint: Show that the governing equations imply $x\dot{x} - y\dot{y} = 0$ and then integrate both sides.)

c) The origin is a saddle point; find equations for its stable and unstable manifolds.

d) The system can be decoupled and solved as follows. Introduce new variables u and v, where $u = x + y$, $v = x - y$. Then rewrite the system in terms of u and v. Solve for $u(t)$ and $v(t)$, starting from an arbitrary initial condition (u_0, v_0).

e) What are the equations for the stable and unstable manifolds in terms of u and v?

f) Finally, using the answer to (d), write the general solution for $x(t)$ and $y(t)$, starting from an initial condition (x_0, y_0).

5.2.1 Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

a) Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of A.

b) Find the general solution of the system.

c) Classify the fixed point at the origin.

d) Solve the system subject to the initial condition $(x_0, y_0) = (3,4)$.

5.2.5 $\dot{x} = 3x - 4y$, $\dot{y} = x - y$

Plot the phase portrait and classify the fixed point of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

5.2.13 (Damped harmonic oscillator) The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

a) Rewrite the equation as a two-dimensional linear system.

b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

6.3.9 Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.

a) Find all the fixed points and classify them.

b) Show that the line $x = y$ is invariant, i.e., any trajectory that starts on it stays on it.

c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all other trajectories. (Hint: Form a differential equation for $x - y$.)

d) Sketch the phase portrait.

e) If you have access to a computer, plot an accurate phase portrait on the square domain $-20 \le x, y \le 20$. (To avoid numerical instability, you'll need to use a fairly small step size, because of the strong cubic nonlinearity.) Notice the trajectories seem to approach a certain curve as $t \to -\infty$; can you explain this behavior intuitively, and perhaps find an approximate equation for this curve?

6.4.9 (Model of a national economy) The following exercise is adapted from Exercise 2.24 in Jordan and Smith (1987). A simple model of a national economy, based on what economists call the "Keynesian cross," is given by $\dot{I} = I - \alpha C$, $\dot{C} = \beta(I - C - G)$, where $I \ge 0$ is the national income, $C \ge 0$ is the rate of consumer spending, and $G \ge 0$ is the rate of government spending. The parameters α and β satisfy $1 < \alpha < \infty$ and $1 \le \beta < \infty$.

a) Show that if the rate of government spending G is *constant*, there is a fixed point for the model, and hence an equilibrium state for the economy. Classify this fixed point as a function of α and β. In the limiting case where $\beta = 1$, show that the economy is predicted to oscillate.

b) Next, assume that government spending increases *linearly* with the national income: $G = G_0 + kI$, where $k > 0$. Determine under what conditions there is an economically sensible equilibrium, meaning one in the first quadrant $I \ge 0$, $C \ge 0$. Show that this kind of equilibrium ceases to exist if k exceeds a critical value k_c, to be determined. How is the economy predicted to behave when $k > k_c$?

c) Finally, suppose government expenditures grow *quadratically* with the national income: $G = G_0 + kI^2$. Show that the system can have two, one, or no fixed points in the first quadrant, depending on how big G_0 is. Discuss the implications for the economy in the various cases by interpreting the phase portraits.