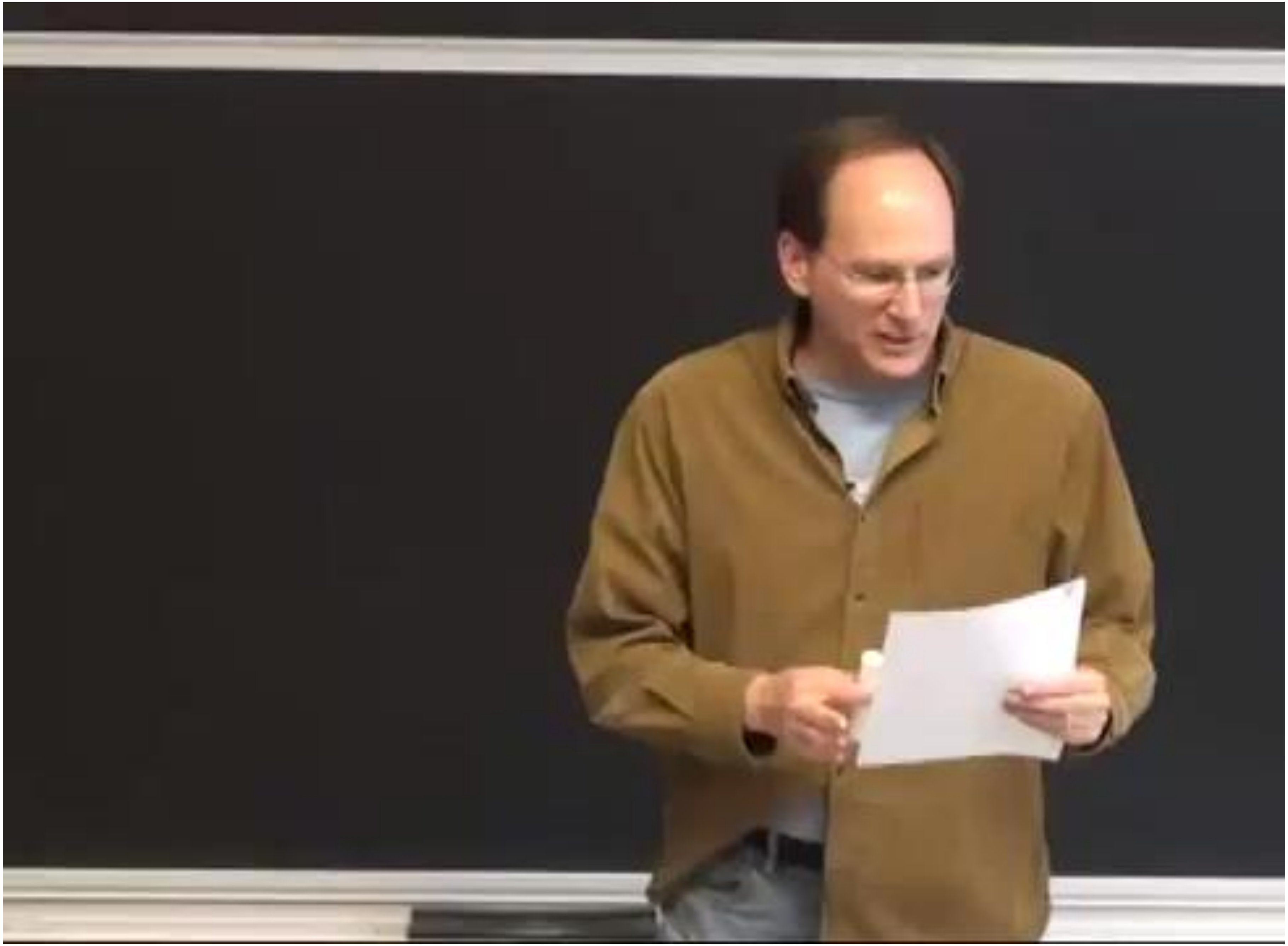
Conservative systems
for self-watch

Lecture 7:

https://www.youtube.com/watch?v=3s2lmZspEU8&list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V&index=7



Main points to pay attention to:

- What does conservation of energy implies
- How the phase space looks like and the idea of centers
- Examples from mechanics, such as a pendulum

Conservative Systems

$$m\ddot{x} = F(x)$$

$$F(x) = -dV/dx.$$

$$m\ddot{x} + \frac{dV}{dx} = 0$$

$$m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0 \Rightarrow \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2 + V(x)\right] = 0$$

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x)$$

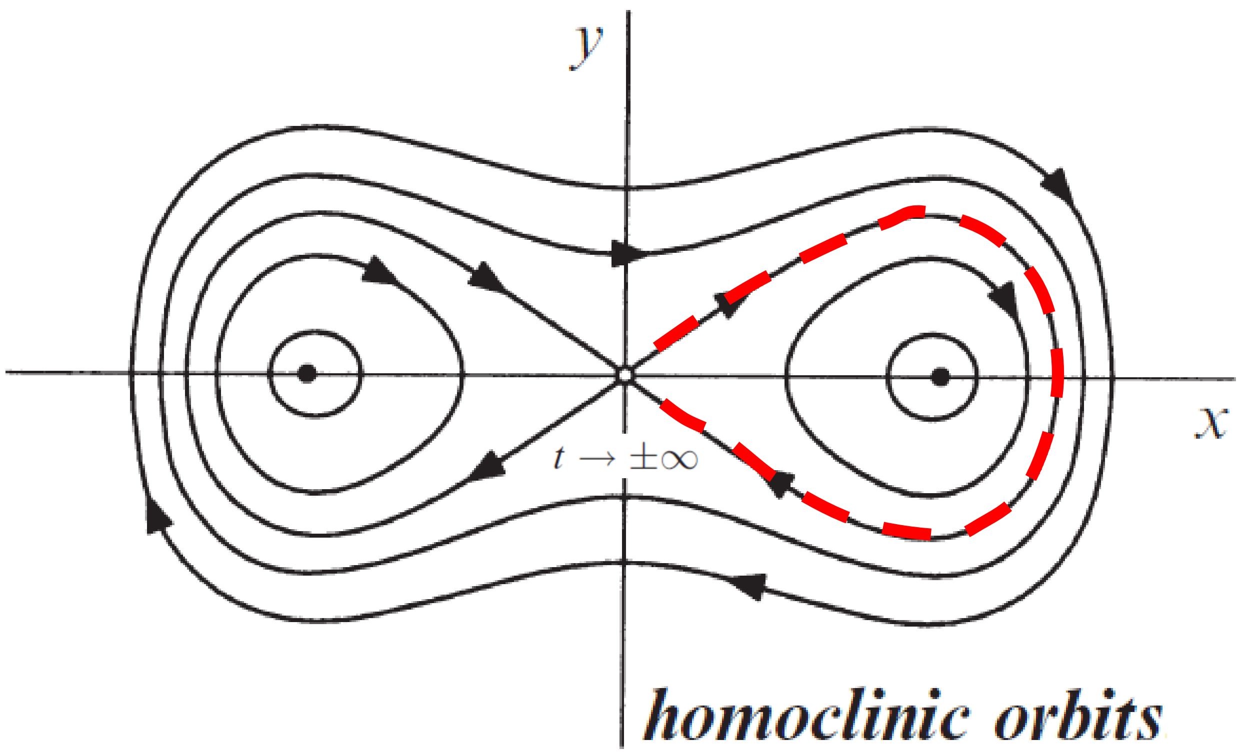
$$dE/dt = 0$$

Consider a particle of mass $m=1$ moving in a double-well potential $V(x)=-\frac{1}{2}x^2+\frac{1}{4}x^4$. Find and classify all the equilibrium points for the system. Then plot the phase portrait and interpret the results physically.

$$\dot{x}=y$$

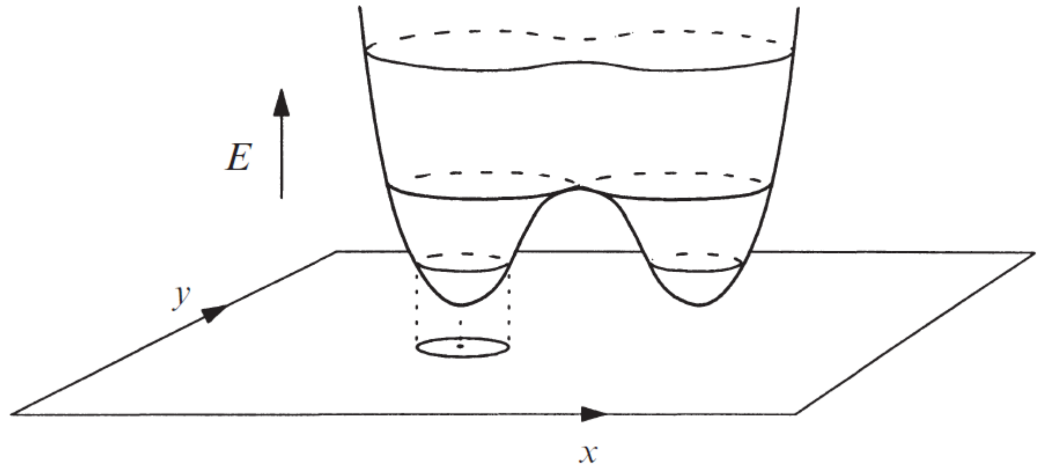
$$\dot{y}=x-x^3$$

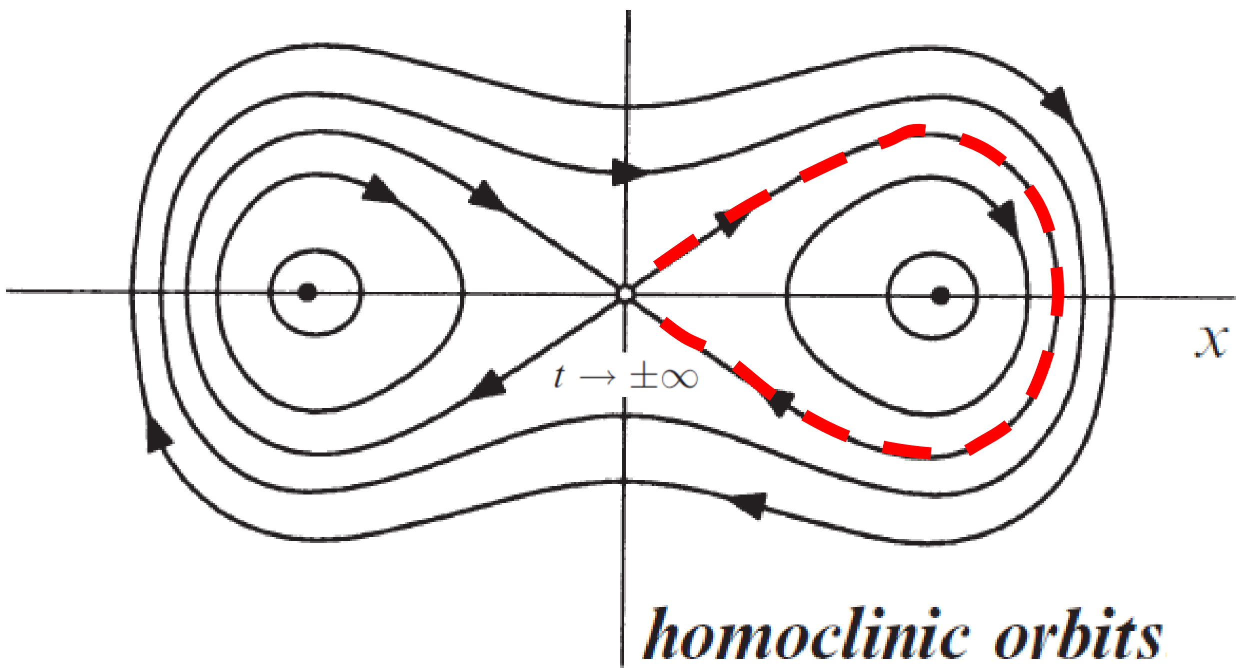
$$E=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{4}x^4=\text{constant}.$$



$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$





Theorem 6.5.1: (Nonlinear centers for conservative systems) Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbf{R}^2$, and $\mathbf{f}$ is continuously differentiable. Suppose there exists a conserved quantity $E(\mathbf{x})$ and suppose that $\mathbf{x}^*$ is an isolated fixed point (i.e., there are no other fixed points in a small neighborhood surrounding $\mathbf{x}^*$). If $\mathbf{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

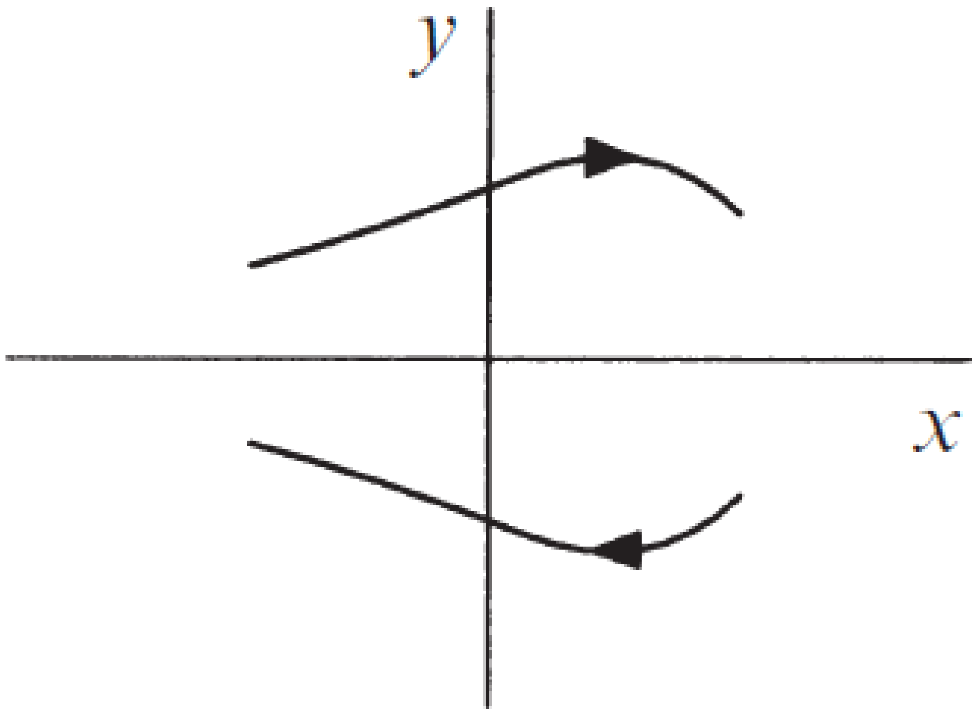
Centers are ordinarily very delicate but, as the examples above suggest, they are much more robust when the system is conservative. We now present a theorem about nonlinear centers in second-order conservative systems.

The theorem says that centers occur at the local minima of the energy function. This is physically plausible—one expects neutrally stable equilibria and small oscillations to occur at the bottom of *any* potential well, no matter what its shape.

Reversible Systems

$$\dot{x} = y$$
$$\dot{y} = \frac{1}{m} F(x)$$

$$t \to -t \text{ and } y \to -y$$



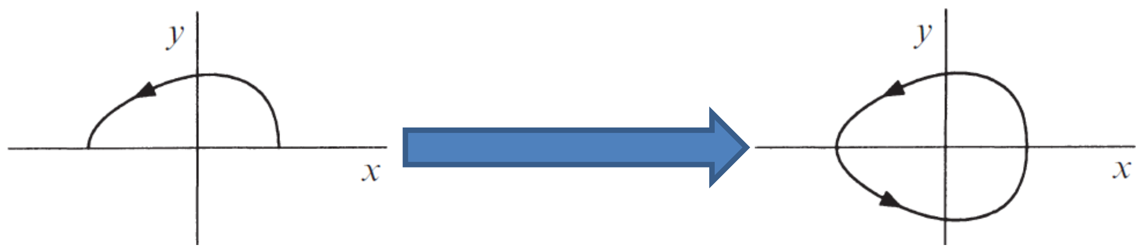
More generally, let's define a ***reversible system*** to be *any* second-order system that is invariant under $t \to -t$ and $y \to -y$. For example, any system of the form

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y),$$

where f is *odd* in y and g is *even* in y (i.e., $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$) is reversible.

Theorem 6.6.1: (Nonlinear centers for reversible systems) Suppose the origin $\mathbf{x}^* = \mathbf{0}$ is a linear center for the continuously differentiable system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y),$$

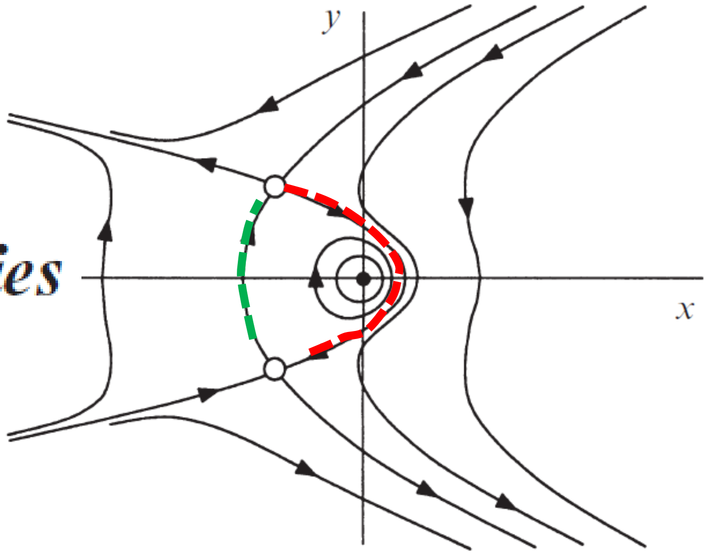


and suppose that the system is reversible. Then sufficiently close to the origin, all trajectories are closed curves.

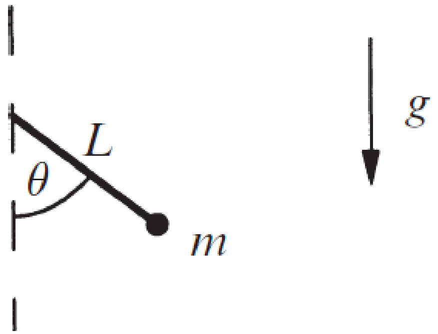
Reversible systems are different from conservative systems, but they have many of the same properties. For instance, the next theorem shows that centers are robust in reversible systems as well.

$$\dot{x} = y - y^3$$
$$\dot{y} = -x - y^2$$

heteroclinic trajectories



Pendulum

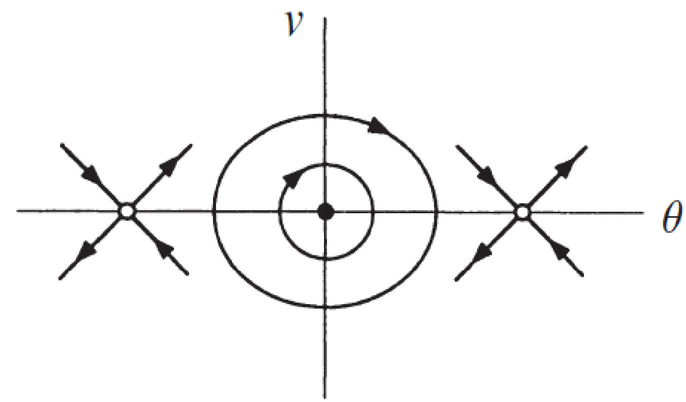


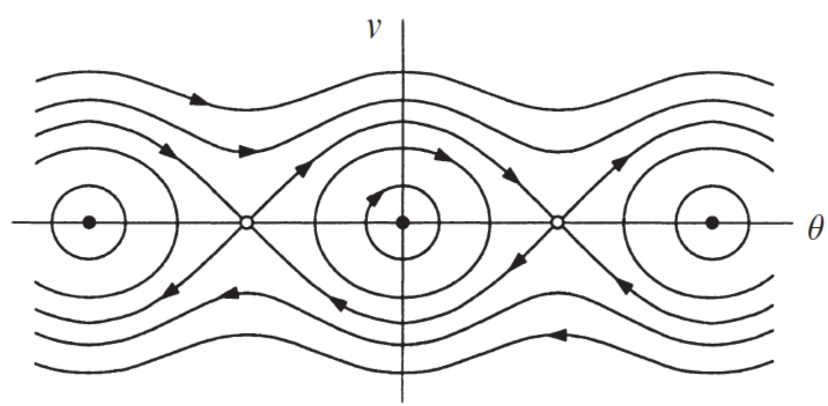
$$\ddot{\theta} + \sin\theta = 0$$

$$\dot{\theta} = v$$

$$\dot{v} = -\sin\theta$$

$$E(\theta, v) = \tfrac{1}{2}v^2 - \cos\theta$$

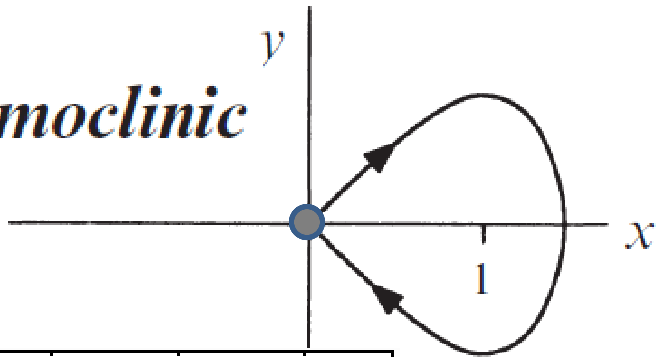


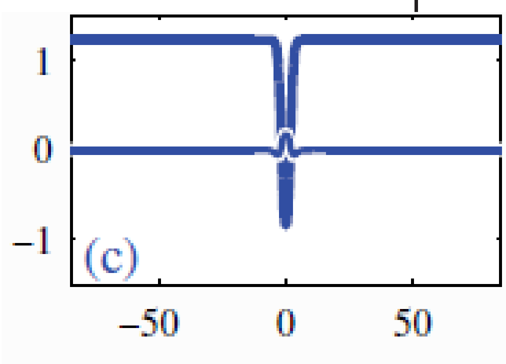


Reversibility

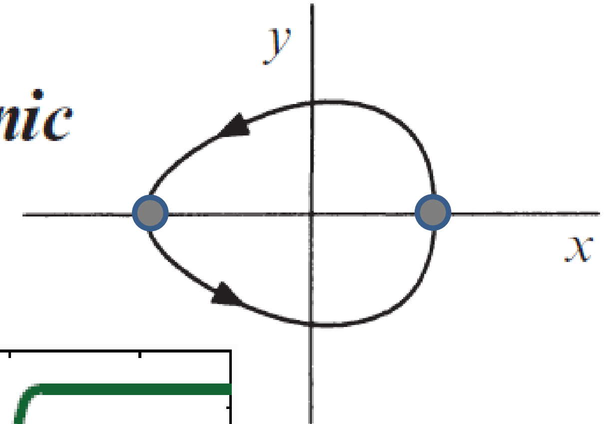
reflection | anti-reflection parity

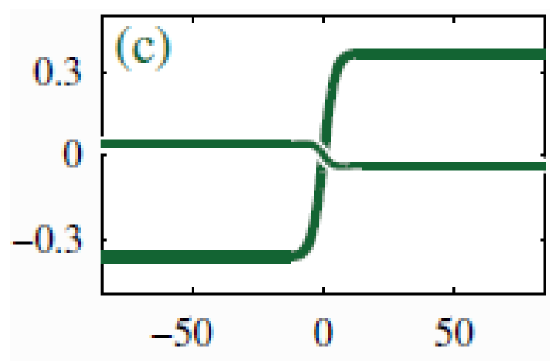
homoclinic

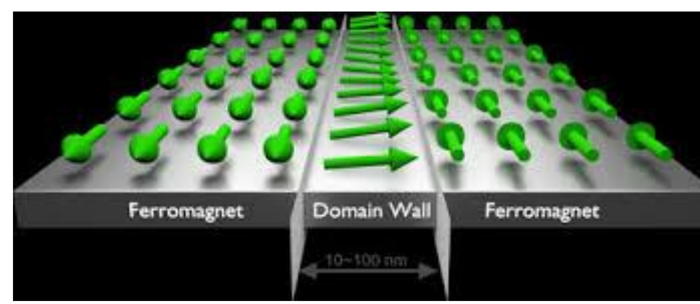




heteroclinic



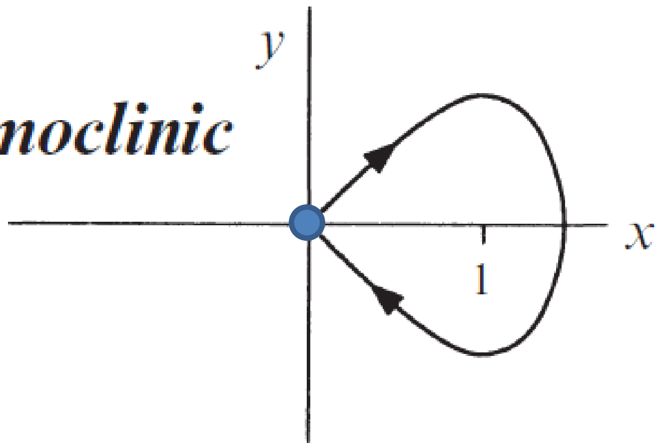




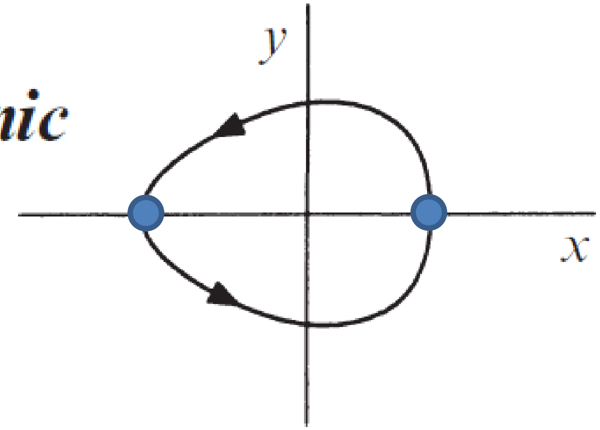
Reversibility

reflection | **anti-reflection parity**

homoclinic



heteroclinic



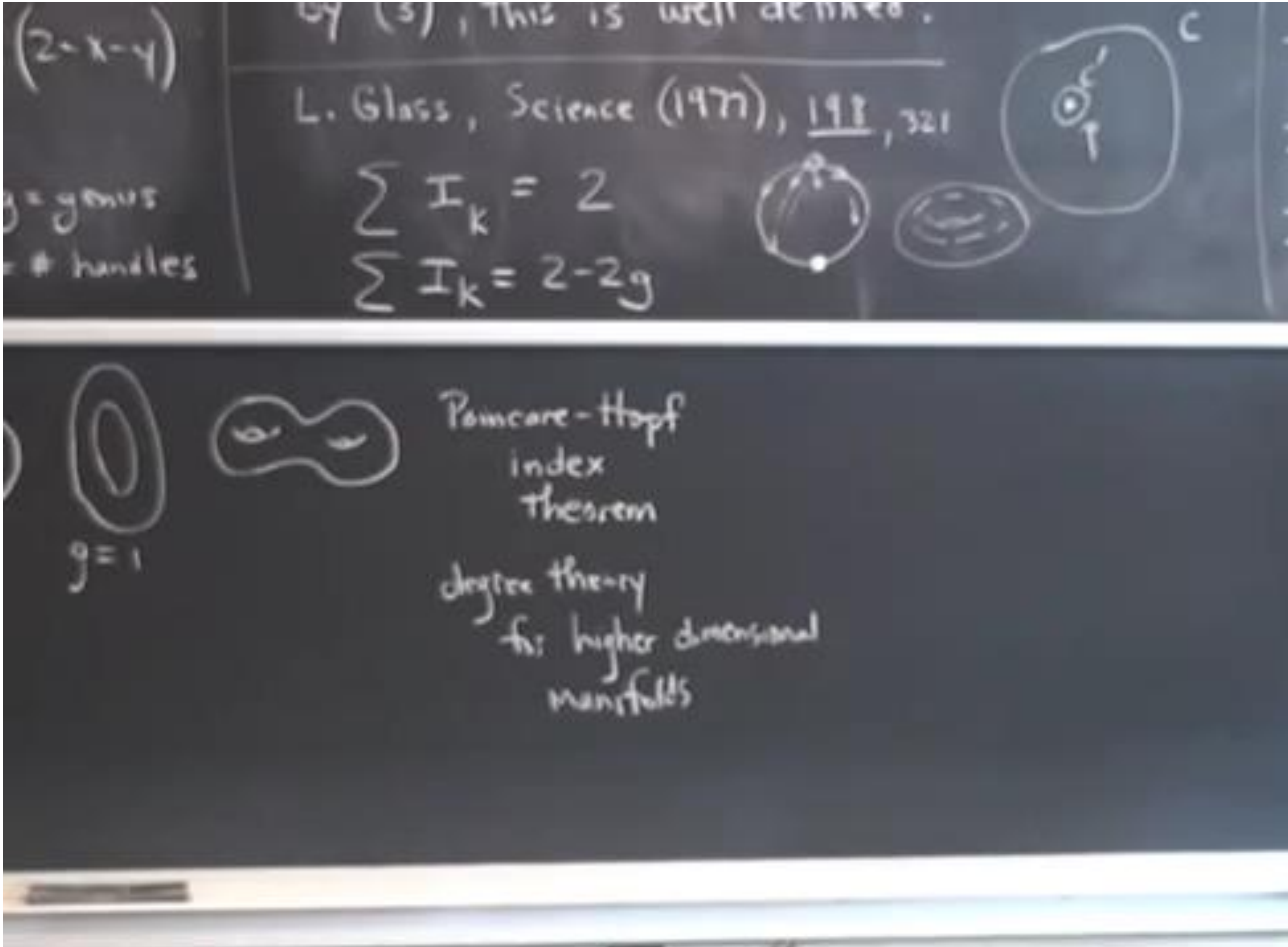
homoclinic and heteroclinic trajectories are <u>**global organizing centers**</u> of the phase space, i.e., GLOBAL BIFURCATIONS

Limit cycles
for self-watch

Lecture 8:

https://www.youtube.com/watch?v=O2fcpxLT5wk&list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V&index=8

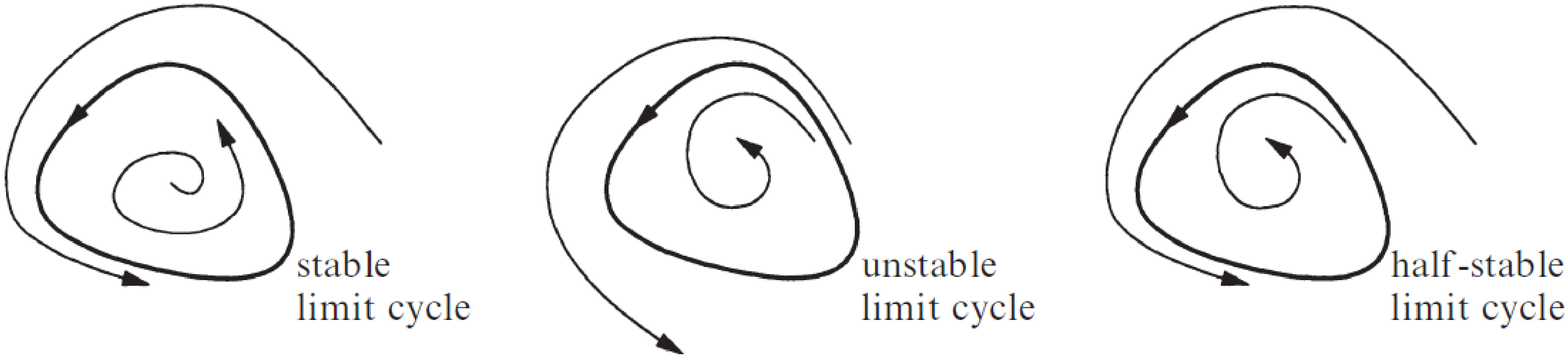
From ~ 1:05:18 (you may skip the index theory BUT if you have time, don't)


(2-x-y)
L. Glass, Science (1977), 198, 321
$\sum I_k = 2$
$\sum I_k = 2-2g$
g = genus
# handles
C
g = 1
Poincare-Hopf
index
theorem
degree theory
for higher dimensional
manifolds

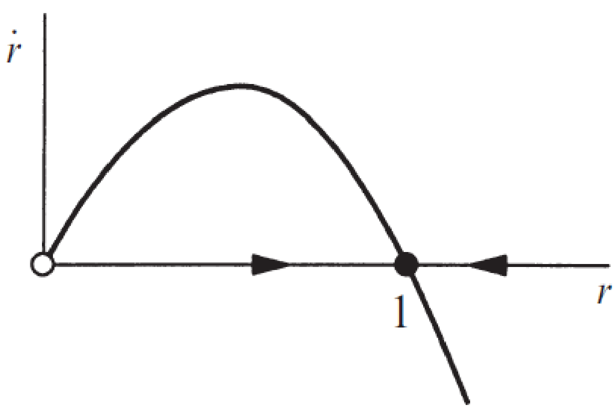
Main points to pay attention to:

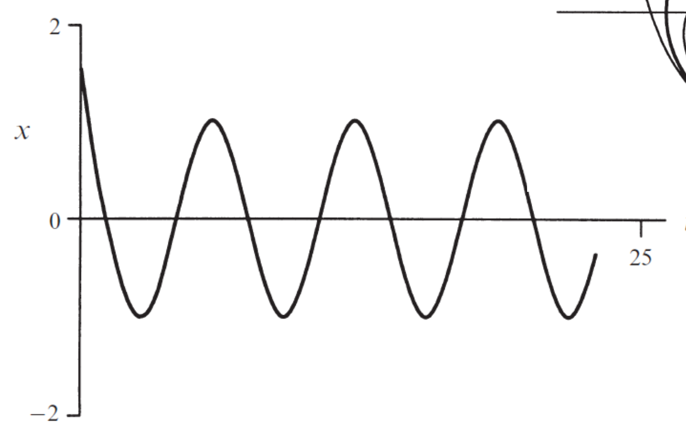
- Periodic orbits: nonlinear oscillations
- At what conditions they may form

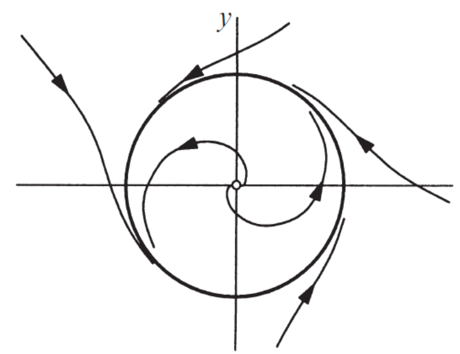
LIMIT CYCLES



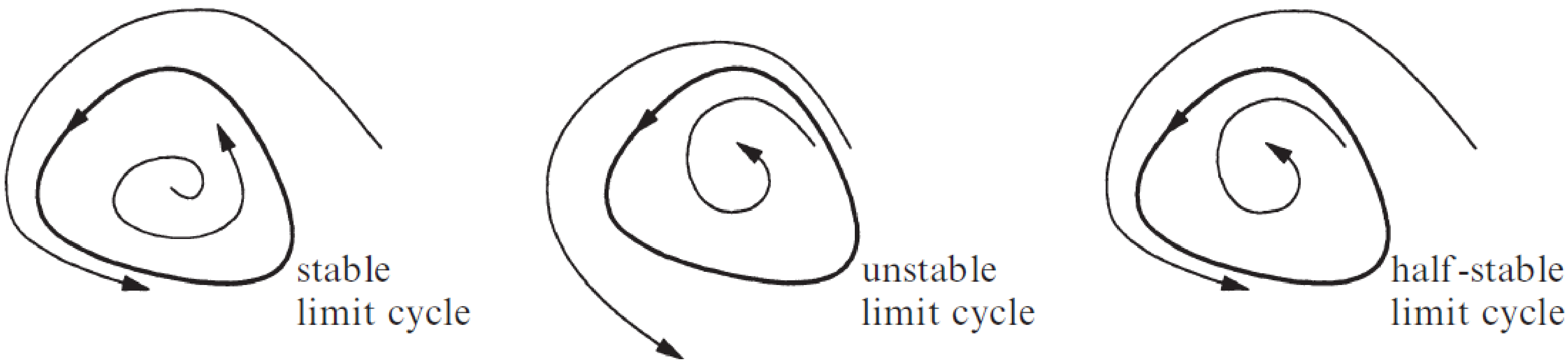
$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

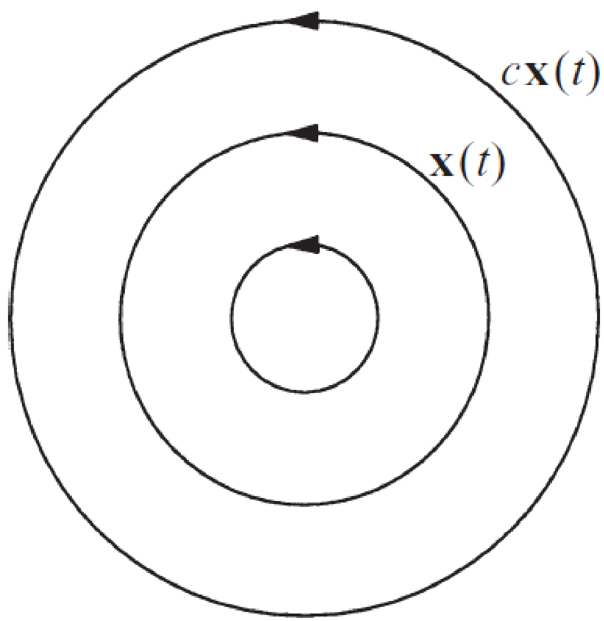


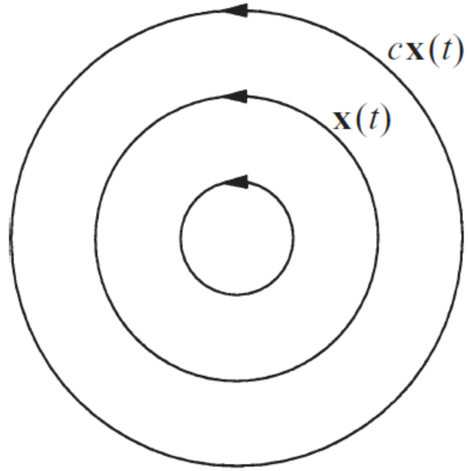




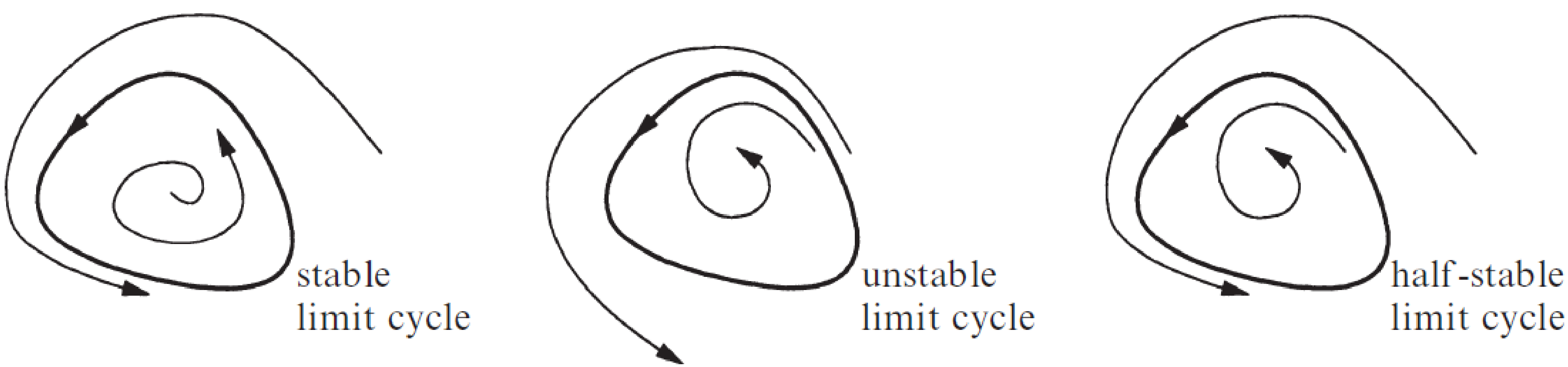
CENTERS AND LIMIT CYCLES







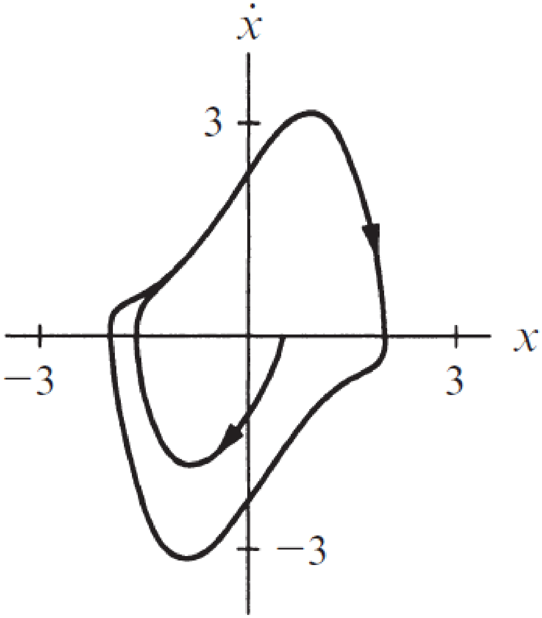
Limit cycles are inherently nonlinear phenomena; they can't occur in linear systems. Of course, a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ can have closed orbits, but they won't be *isolated*; if $\mathbf{x}(t)$ is a periodic solution, then so is $c\mathbf{x}(t)$ for any constant $c \neq 0$. Hence $\mathbf{x}(t)$ is surrounded by a one-parameter family of closed orbits (Figure 7.0.2).

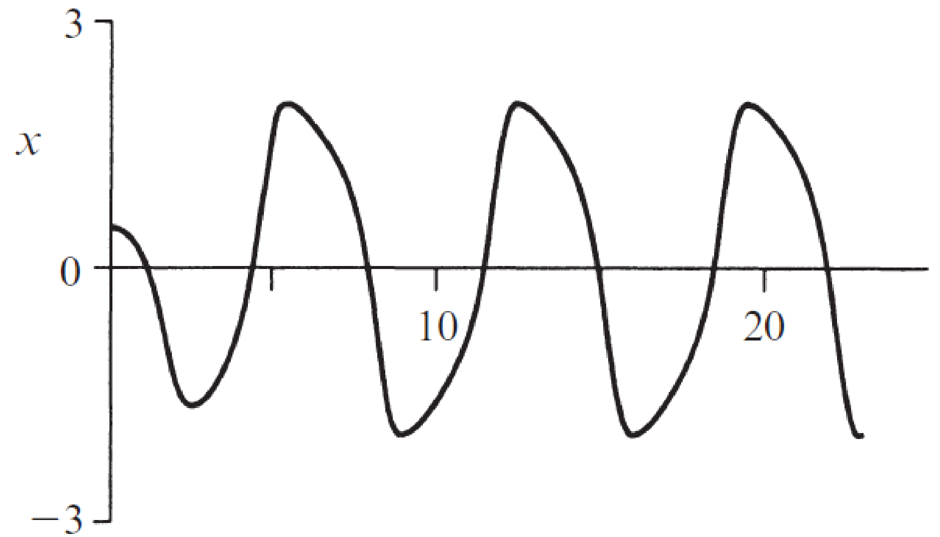


VAN DER POL OSCILLATOR

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

damping

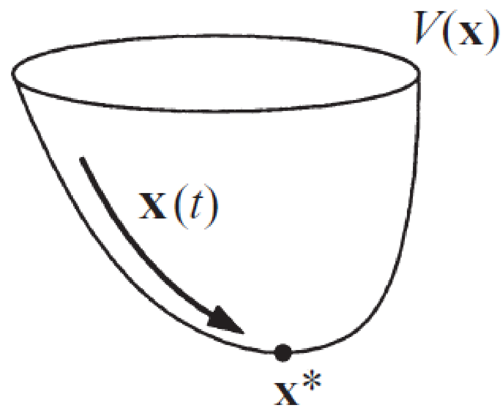




Gradient Systems

Suppose the system can be written in the form $\dot{\mathbf{x}} = -\nabla V$, for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$. Such a system is called a ***gradient system*** with ***potential function*** V.

Theorem 7.2.1: Closed orbits are impossible in gradient systems.



To be more precise, consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose that we can find a ***Liapunov function***, i.e., a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

1. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite*.)
2. $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward $\mathbf{x}^*$.)