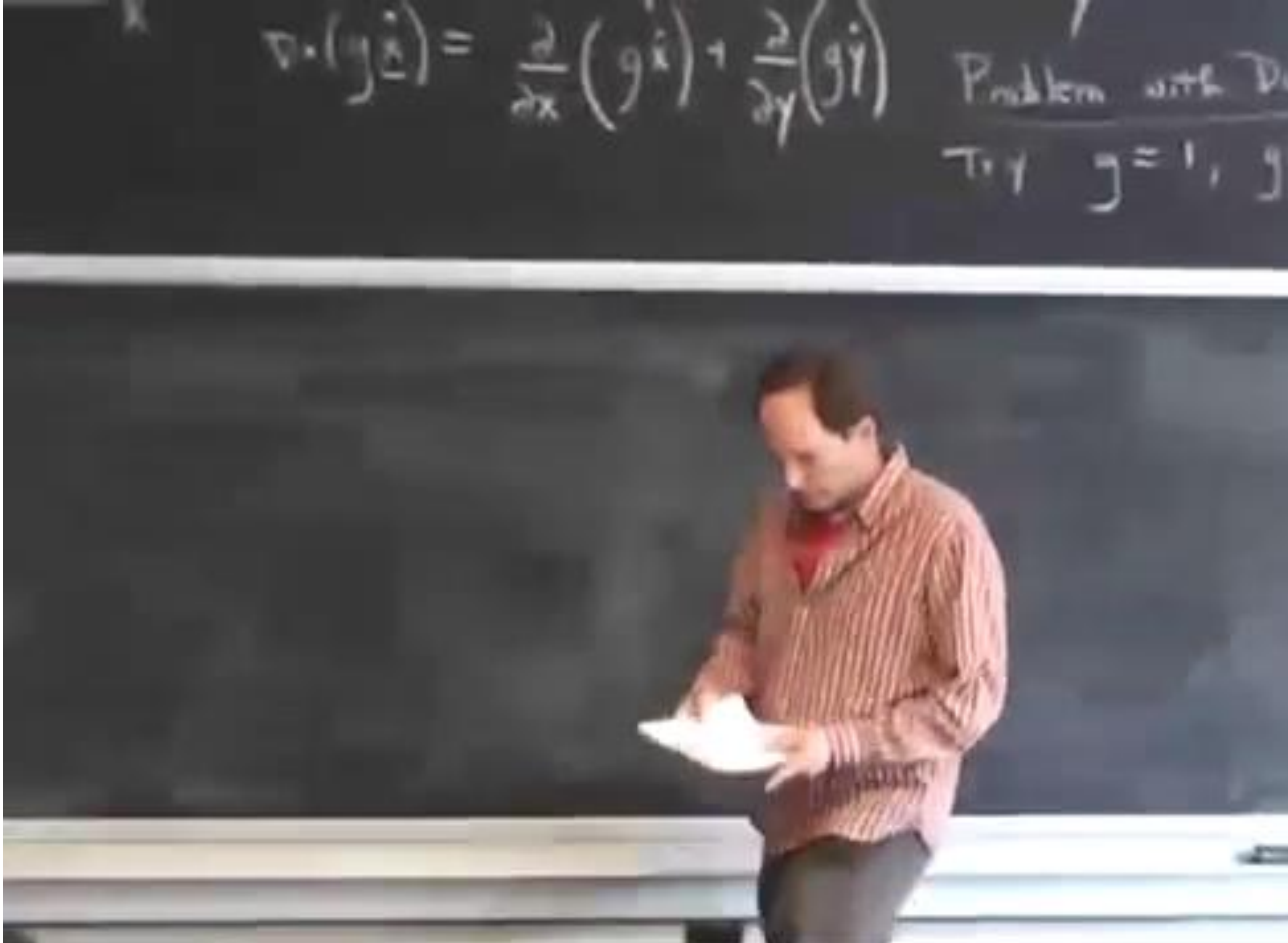
Properties of closed orbits

for self-watch

Lecture 9:

https://www.youtube.com/watch?v=nWO74rlr58Y&list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V&index=9

From ~ 28:22 (skip Dulac's criterion)


$\nabla \cdot (g \dot{\underline{x}}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$
Problem with
Try $g = 1$,

Main points to pay attention to:

- Poincaré-Bendixson theorem: existence of periodic orbits
- Nullclines

PROPERTIES OF CLOSED ORBITS

Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time (Figure 7.3.1).

Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, *R contains a closed orbit* (shown as a heavy curve in Figure 7.3.1).

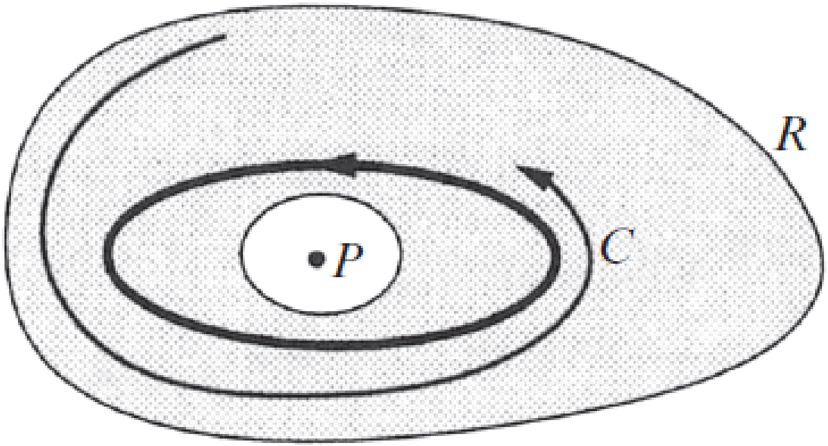


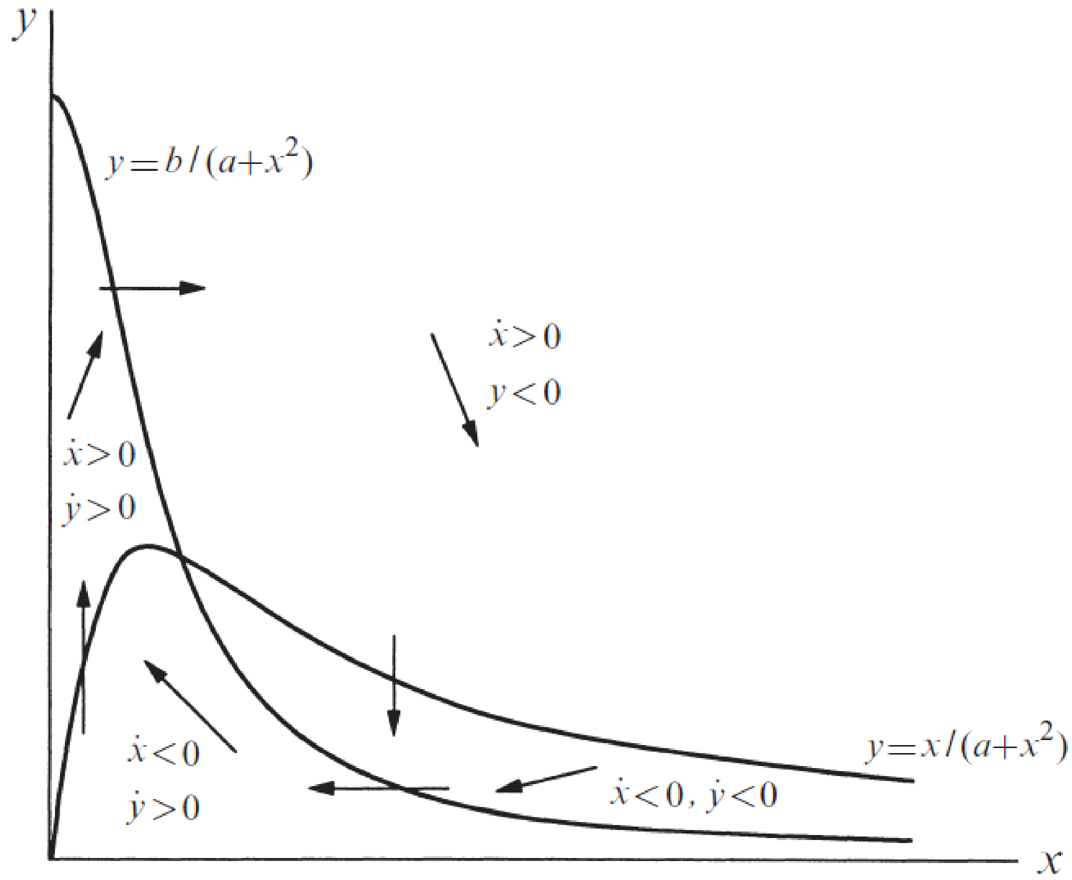
Figure 7.3.1

glycolysis

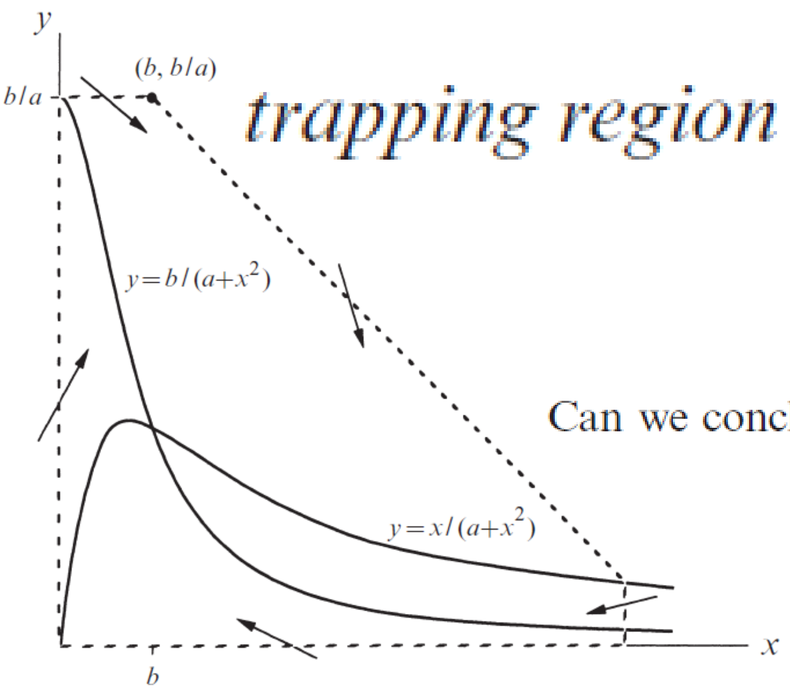
$$\dot{x} = -x + ay + x^2 y$$

$$\dot{y} = b - ay - x^2 y$$

$a,b > 0$ are kinetic parameters



trapping region



Can we conclude that there is a closed orbit inside the trapping region?

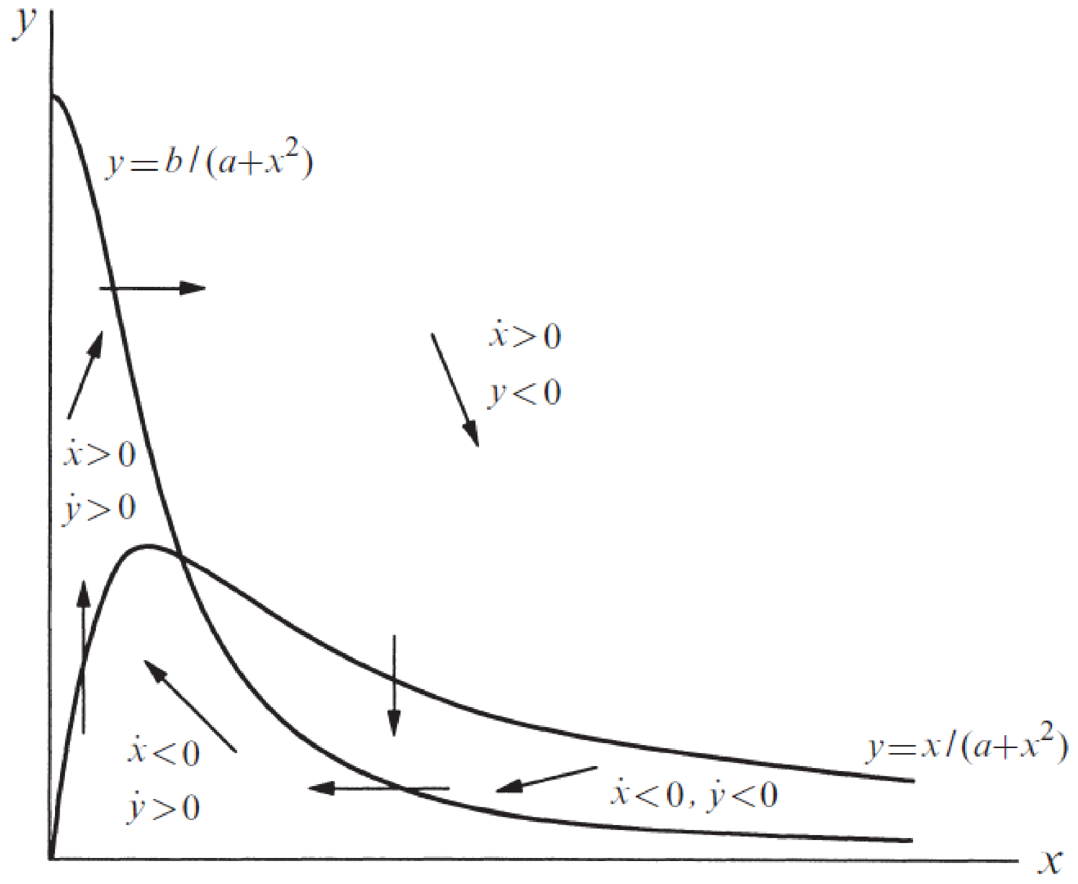
NO! WHY?

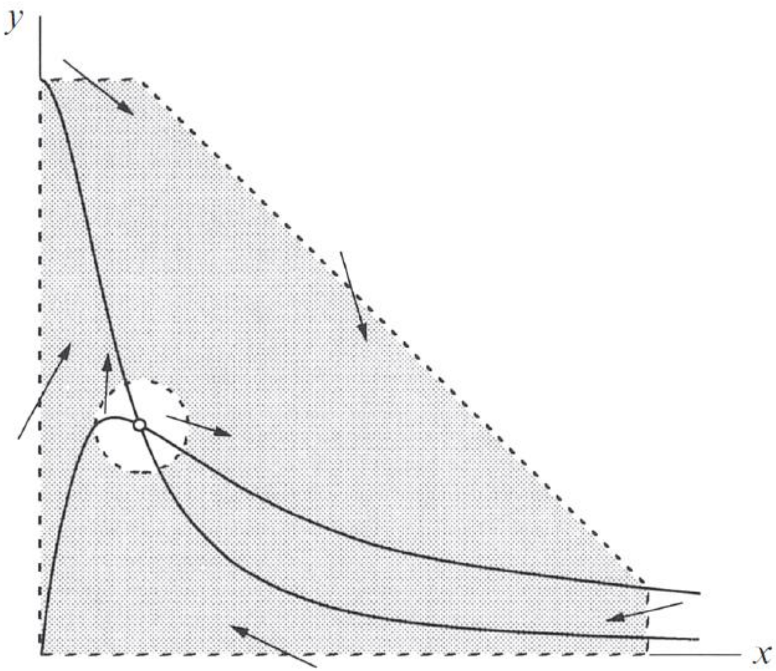
glycolysis

$$\dot{x} = -x + ay + x^2 y$$

$$\dot{y} = b - ay - x^2 y$$

$a,b > 0$ are kinetic parameters





if this fixed point is a *repeller*, then it is possible to prove the existence of a closed orbit (we won't do it)

Show that a closed orbit exists if *a* and *b* satisfy an appropriate condition

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

$a, b > 0$ are kinetic parameters

It suffices to find **conditions** under which the **fixed point is a repeller**, i.e., an unstable node or spiral

- Jacobian $$A = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{pmatrix}$$

- Fixed point $$x^* = b, \qquad y^* = \frac{b}{a + b^2},$$

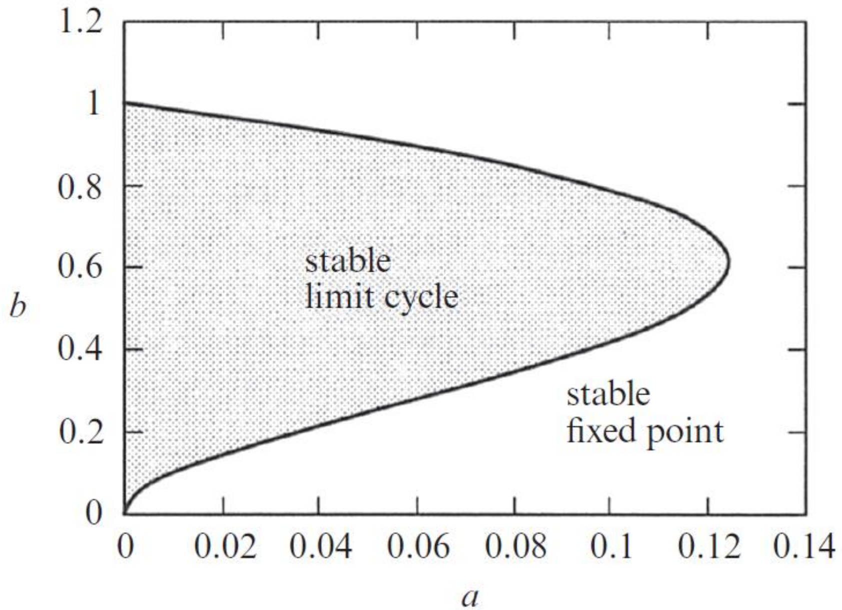
- Check stability

$$\Delta = a + b^2 > 0 \qquad \tau = -\frac{b^4 + (2a-1)b^2 + (a+a^2)}{a+b^2}$$

√ ?

Hence the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$

- We obtain the condition: $b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right)$



The Poincare-Bendixson theorem is a central result.

It says that the **dynamical possibilities in the phase plane are very limited**: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit

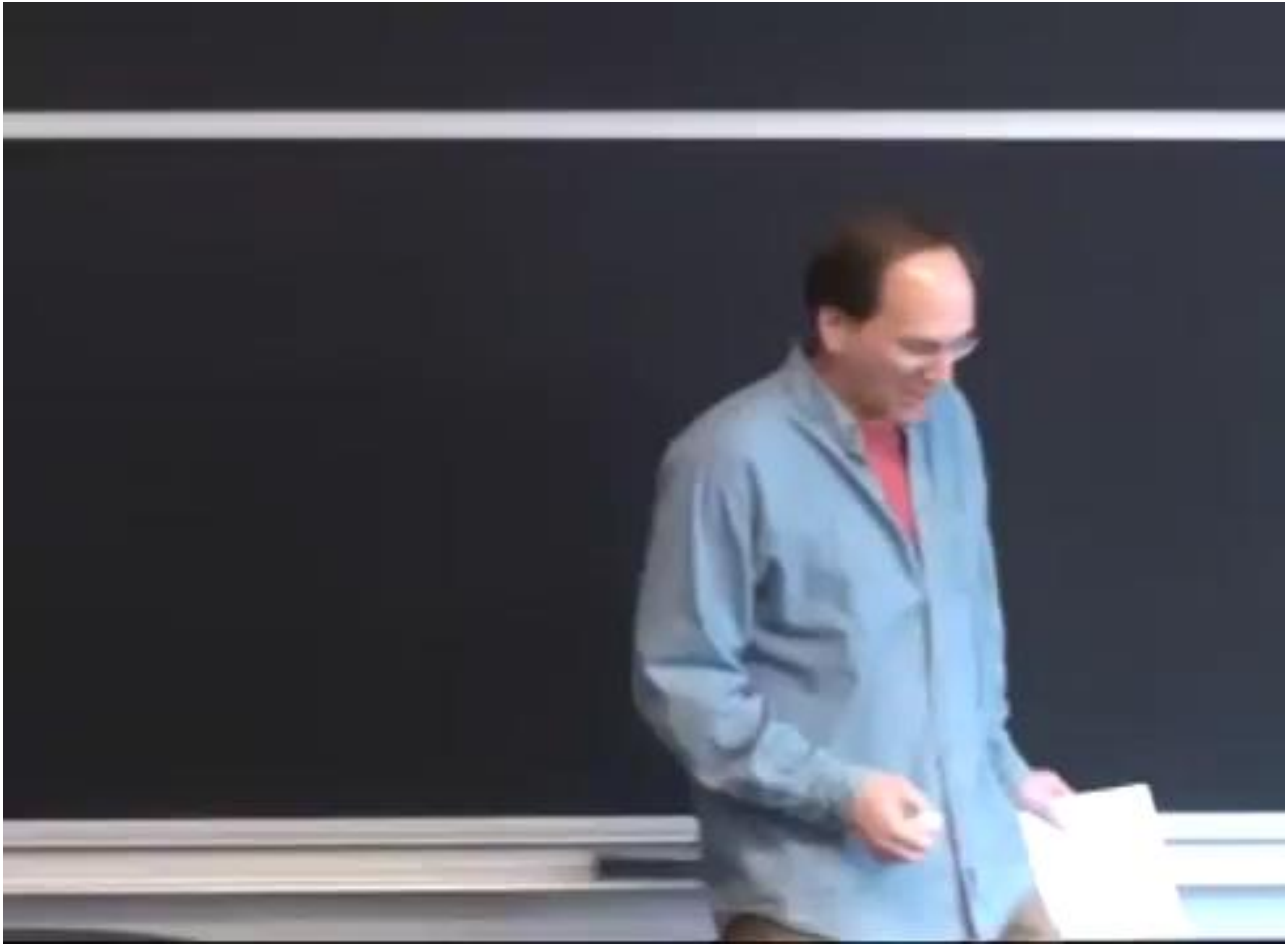
$\Rightarrow$ NO CHAOS

This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n > 3$), the Poincare-Bendixson theorem no longer applies

Relaxation oscillations
for self-watch

Lecture 10:

https://www.youtube.com/watch?v=O1lQrHemPsw&list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V&index=10



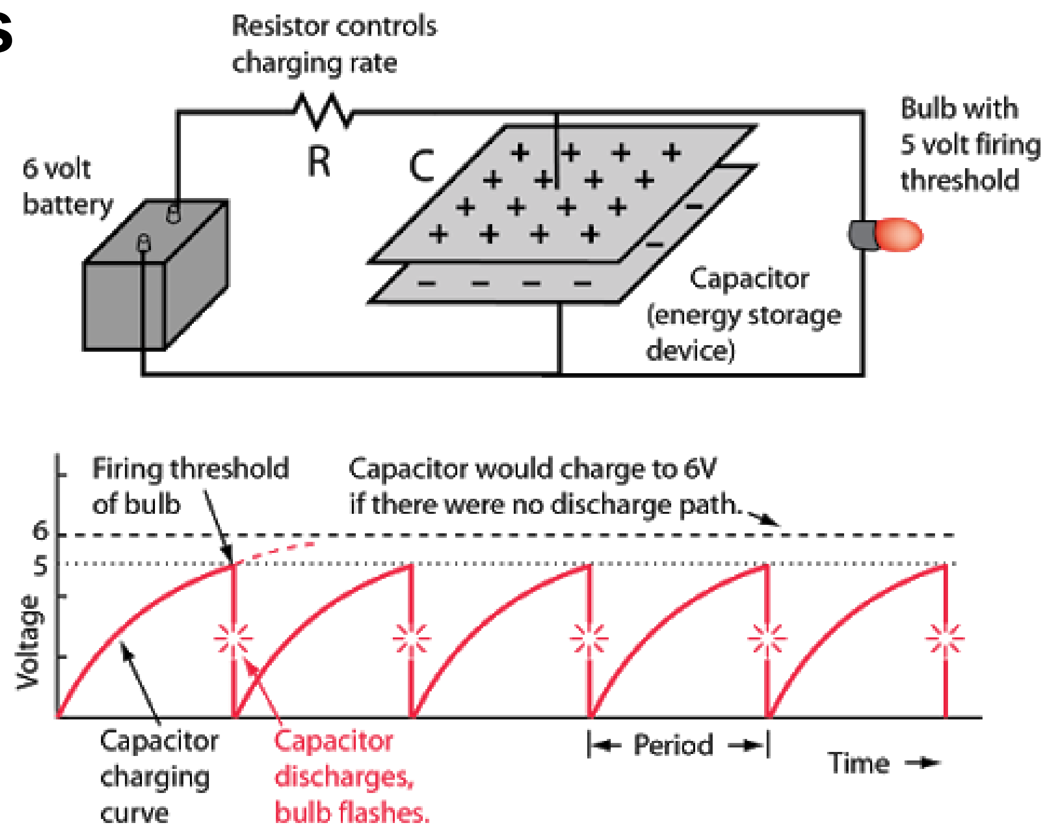
Main points to pay attention to:

- Properties of nonlinear oscillations
- How to approximate different limits of the dynamics
- How to use energy to estimate amplitude of the oscillations

Relaxation Oscillations

Oscillations where the "stress" accumulated during the slow buildup is "relaxed" during the sudden discharge

…in other words: periodic solutions that exhibit **fast and slow dynamical features**



Appear in almost all media

- Physical: EM, optics
- Biological: neurons, heart
- Chemistry: autocatalysis
- Population dynamics …

Example: van der Pol equation $\ddot{x}+\mu(x^2-1)\dot{x}+x=0$

$\mu >> 1$

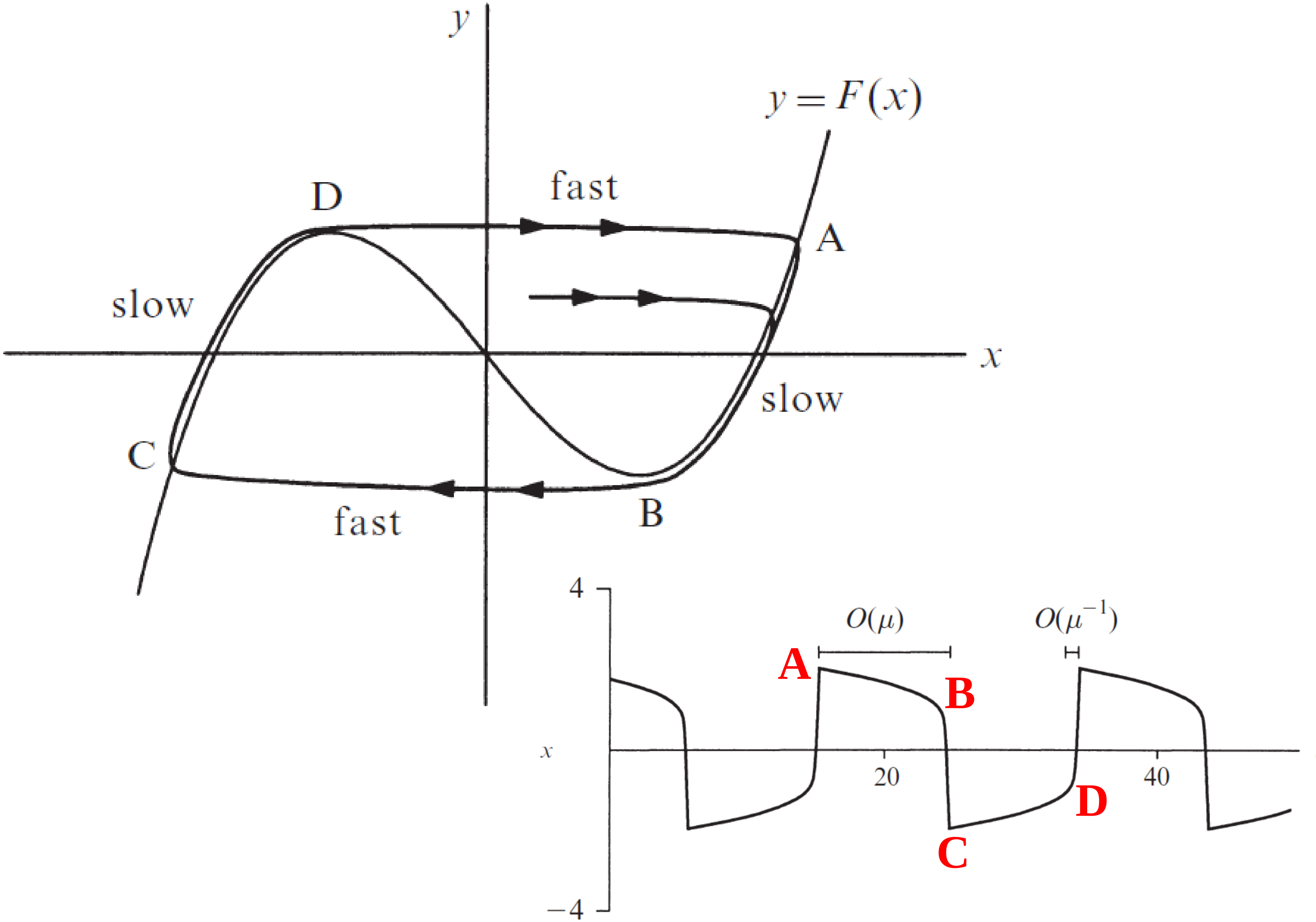
$$\ddot{x}+\mu\dot{x}(x^2-1)=\frac{d}{dt}\left(\dot{x}+\mu\left[\tfrac{1}{3}x^3-x\right]\right).$$

So if we let

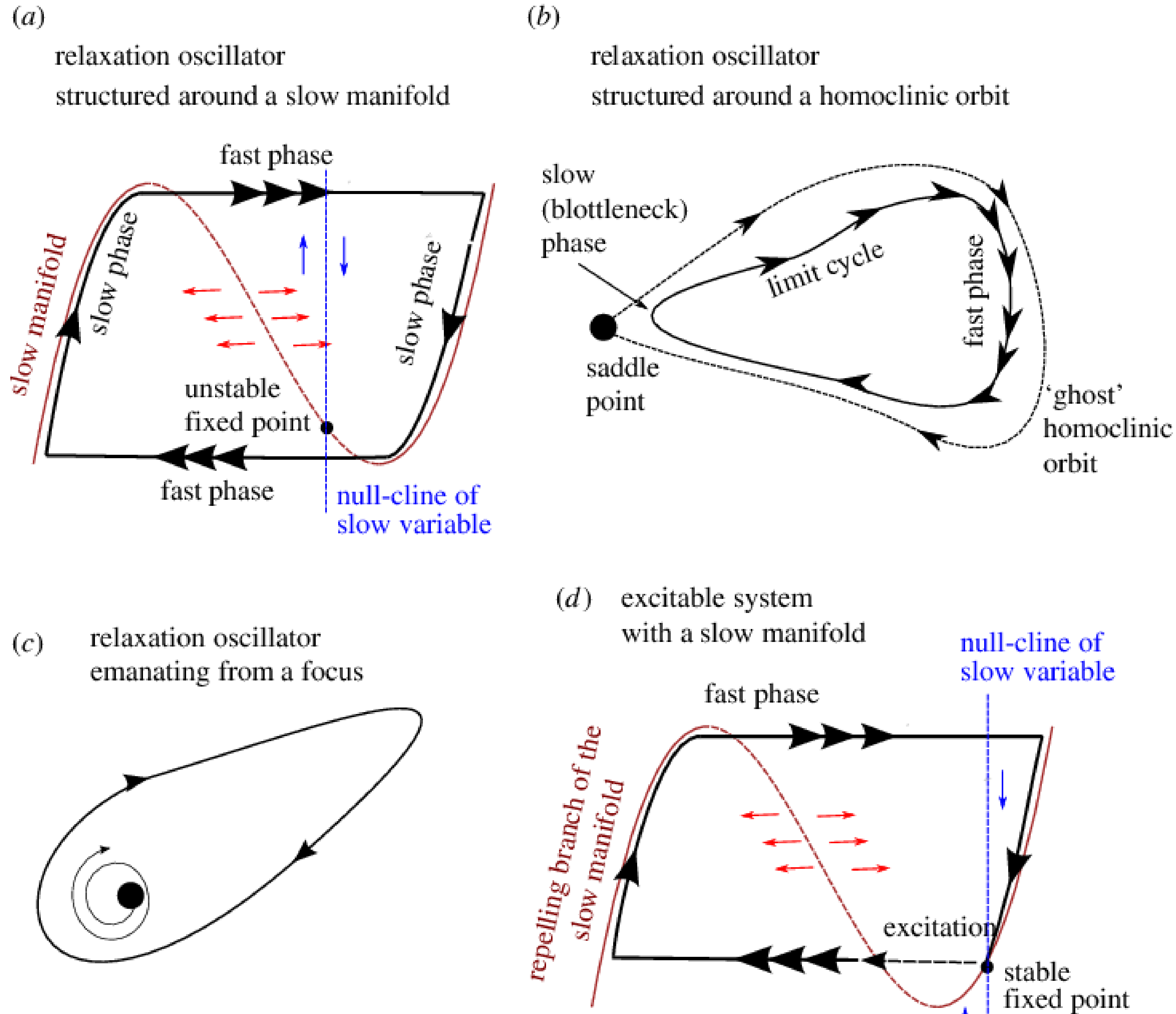
$$F(x)=\tfrac{1}{3}x^3-x, \qquad w=\dot{x}+\mu F(x), \qquad y=\frac{w}{\mu}$$

the van der Pol equation implies that

$$\boxed{\begin{aligned}\dot{x}&=\mu\left[y-F(x)\right]\\ \dot{y}&=-\frac{1}{\mu}x.\end{aligned}}$$



y
y = F(x)
fast
D
A
slow
x
slow
C
fast
B
4
O(μ)
O(μ⁻¹)
A
B
x
t
20
40
D
C
−4


(a)
relaxation oscillator
structured around a slow manifold
fast phase
slow manifold
slow phase
slow phase
unstable
fixed point
fast phase
null-cline of
slow variable
(b)
relaxation oscillator
structured around a homoclinic orbit
slow
(blottleneck)
phase
limit cycle
fast phase
saddle
point
'ghost'
homoclinic
orbit
(c)
relaxation oscillator
emanating from a focus
(d)
excitable system
with a slow manifold
null-cline of
slow variable
fast phase
repelling branch of the
slow manifold
excitation
stable
fixed point