Suggested home exercises – lecture 6

7.3.4 Consider the system

$$\dot{x} = x(1-4x^2-y^2) - \tfrac{1}{2}y(1+x), \qquad \dot{y} = y(1-4x^2-y^2) + 2x(1+x).$$

a) Show that the origin is an unstable fixed point.
b) By considering $\dot{V}$, where $V = (1-4x^2-y^2)^2$, show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$.

7.3.5 Show that the system $\dot{x} = -x - y + x(x^2+2y^2)$, $\dot{y} = x - y + y(x^2+2y^2)$ has at least one periodic solution.

7.3.11 (Cycle graphs) Suppose $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a smooth vector field on $\mathbf{R}^2$. An improved version of the Poincaré-Bendixson theorem states that if a trajectory is trapped in a compact region, then it must approach a fixed point, a closed orbit, or something exotic called a ***cycle graph*** (an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented either clockwise or counterclockwise). Cycle graphs are rare in practice; here's a contrived but simple example.
a) Plot the phase portrait for the system

$$\dot{r} = r(1-r^2)\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$$
$$\dot{\theta} = r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2$$

where r,θ are polar coordinates. (Hint: Note the common factor in the two equations; examine where it vanishes.)
b) Sketch x vs. t for a trajectory starting away from the unit circle. What happens as $t \to \infty$?

7.5.1 For the van der Pol oscillator with $\mu >> 1$, show that the positive branch of the cubic nullcline begins at $x_A = 2$ and ends at $x_B = 1$.

7.5.7 (Cell cycle) Tyson (1991) proposed an elegant model of the cell division cycle, based on interactions between the proteins cdc2 and cyclin. He showed that the model's mathematical essence is contained in the following set of dimensionless equations:

$$\dot{u} = b(v-u)(\alpha+u^2)-u, \qquad \dot{v} = c-u,$$

where u is proportional to the concentration of the active form of a cdc2-cyclin complex, and v is proportional to the total cyclin concentration (monomers and dimers). The parameters $b >> 1$ and $\alpha << 1$ are fixed and satisfy $8\alpha b < 1$, and c is adjustable.

a) Sketch the nullclines.

b) Show that the system exhibits relaxation oscillations for $c_1 < c < c_2$, where c_1 and c_2 are to be determined approximately. (It is too hard to find c_1 and c_2 exactly, but a good approximation can be achieved if you assume $8\alpha b << 1$.)

c) Show that the system is excitable if c is slightly less than c_1.