Suggested home exercises – lecture 7

8.1.7 Find and classify all bifurcations for the system $\dot{x} = y - ax$, $\dot{y} = -by + x/(1+x)$.

8.1.9 Plot the stability diagram for the system $\ddot{x} + b\dot{x} - kx + x^3 = 0$, where b and k can be positive, negative, or zero. Label the bifurcation curves in the (b, k) plane.

8.1.13 (Laser model) In Exercise 3.3.1 we introduced the laser model

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p$$

where $N(t)$ is the number of excited atoms and $n(t)$ is the number of photons in the laser field. The parameter G is the gain coefficient for stimulated emission, k is the decay rate due to loss of photons by mirror transmission, scattering, etc., f is the decay rate for spontaneous emission, and p is the pump strength. All parameters are positive, except p, which can have either sign. For more information, see Milonni and Eberly (1988).

8.2.1 Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in (μ, a) space at which Hopf bifurcations occur.

8.2.9 Consider the predator-prey model

$$\dot{x} = x\left(b - x - \frac{y}{1+x}\right), \quad \dot{y} = y\left(\frac{x}{1+x} - ay\right),$$

where $x, y \geq 0$ are the populations and $a, b > 0$ are parameters.

a) Sketch the nullclines and discuss the bifurcations that occur as b varies.

b) Show that a positive fixed point $x^* > 0$, $y^* > 0$ exists for all $a, b > 0$. (Don't try to find the fixed point explicitly; use a graphical argument instead.)

c) Show that a Hopf bifurcation occurs at the positive fixed point if

$$a = a_c = \frac{4(b-2)}{b^2(b+2)}$$

and $b > 2$. (Hint: A necessary condition for a Hopf bifurcation to occur is $\tau = 0$, where τ is the trace of the Jacobian matrix at the fixed point. Show that $\tau = 0$ if and only if $2x^* = b - 2$. Then use the fixed point conditions to express a_c in terms of x^*. Finally, substitute $x^* = (b - 2)/2$ into the expression for a_c and you're done.)

d) Using a computer, check the validity of the expression in (c) and determine whether the bifurcation is subcritical or supercritical. Plot typical phase portraits above and below the Hopf bifurcation.