Suggested home exercises – lecture 9

9.2.1 (Parameter where Hopf bifurcation occurs)

a) For the Lorenz equations, show that the characteristic equation for the eigenvalues of the Jacobian matrix at C^+, C^- is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0.$$

b) By seeking solutions of the form $\lambda = i\omega$, where ω is real, show that there is a pair of pure imaginary eigenvalues when $r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)$. Explain why we need to assume $\sigma > b + 1$.

c) Find the third eigenvalue.

9.2.3 (A spherical trapping region) Show that all trajectories eventually enter and remain inside a large sphere S of the form $x^2 + y^2 + (z - r - \sigma)^2 = C$, for C sufficiently large. (Hint: Show that $x^2 + y^2 + (z - r - \sigma)^2$ decreases along trajectories for all (x, y, z) outside a certain fixed ellipsoid. Then pick C large enough so that the sphere S encloses this ellipsoid.)

(Numerical experiments) For each of the values of r given below, use a computer to explore the dynamics of the Lorenz system, assuming $\sigma = 10$ and $b = 8/3$ as usual. In each case, plot $x(t)$, $y(t)$, and x vs. z. You should investigate the consequences of choosing different initial conditions and lengths of integration. Also, in some cases you may want to ignore the transient behavior, and plot only the sustained long-term behavior.

9.3.2 $r = 10$

9.3.3 $r = 22$ (transient chaos)

9.3.4 $r = 24.5$ (chaos and stable point co-exist)

9.3.5 $r = 100$ (surprise)

9.3.6 $r = 126.52$

9.3.7 $r = 400$