(a) Our wavefunction is given by

$$\psi = C\left(\phi_{100} + 4i\phi_{210} - 2\sqrt{2}\phi_{22-1}\right)$$

To normalize, we compute $(\phi|\phi)$ and set it to 1:

$$\begin{aligned}
(\psi|\psi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C^*\left(\phi_{100} + 4i\phi_{210} - 2\sqrt{2}\phi_{22-1}\right)^* \times \\
&\qquad C\left(\phi_{100} + 4i\phi_{210} - 2\sqrt{2}\phi_{22-1}\right) d^3\vec{r} \\
&= |C|^2\Bigg(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{100}^*\phi_{100} d^3\vec{r} + \\
&\qquad +16\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{210}^*\phi_{210} d^3\vec{r} + 8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{22-1}^*\phi_{22-1} d^3\vec{r}\Bigg) \\
&= |C|^2\left[(\phi_{100}|\phi_{100}) + 16(\phi_{210}|\phi_{210}) + 8(\phi_{22-1}|\phi_{22-1})\right] \\
&= |C|^2(1 + 16 + 8) = 25|C|^2,
\end{aligned}$$

where we have used the fact that the energy eigenfunctions are normalized. Note that "cross-terms" do not appear because the energy eigenstates are orthogonal. Choosing C to be real gives

$$C = \frac{1}{5}.$$

(b) Finding the expectation value of $\hat{E}$ requires very similar manipulations, except this time we need to "sandwich" the energy operator in between the two copies of ψ:

$$\langle\hat{E}\rangle = (\psi|\hat{E}\psi).$$

Note that the cross-terms *still* do not appear, because the energy eigenstates satisfy the energy eigenvalue equation *i.e.*

$$\hat{E}\psi_{nlm} = E_n\psi_{nlm},$$

so for example

$$(\psi_{100}|\hat{E}\psi_{311}) = E_3(\psi_{100}|\psi_{311}) = 0.$$

Armed with this, we can immediately say

$$\begin{aligned}
\hat{E} &= (\psi|\hat{E}\psi) \\
&= |C|^2\left[(\phi_{100}|\hat{E}\phi_{100}) + 16(\phi_{210}|\hat{E}\phi_{210}) + 8(\phi_{22-1}|\hat{E}\phi_{22-1})\right] \\
&= |C|^2\left[E_1(\phi_{100}|\phi_{100}) + 16E_2(\phi_{210}|\phi_{210}) + 8E_2(\phi_{22-1}|\phi_{22-1})\right] \\
&= \frac{1}{51}(E_1 + 24E_2).
\end{aligned}$$

(c) The algebra required to find $\hat{L}^2$ is very similar to what we have already done:

$$\begin{aligned}
\hat{L}^2 &= (\psi|\hat{L}^2\psi) \\
&= |C|^2[(\phi_{100}|\hat{L}^2\phi_{100}) + 16(\phi_{210}|\hat{L}^2\phi_{210}) + 8(\phi_{22-1}|\hat{L}^2\phi_{22-1})] \\
&= |C|^2[(0\hbar^2)(\phi_{100}|\phi_{100}) + 16(2\hbar^2)(\phi_{210}|\phi_{210}) + 8(6\hbar^2)(\phi_{22-1}|\phi_{22-1})] \\
&= \frac{80\hbar^2}{25}.
\end{aligned}$$

(d) Finally, for $\hat{L}_z$ we have

$$\begin{aligned}
\hat{L}_z &= (\psi|\hat{L}_z\psi) \\
&= |C|^2[(\phi_{100}|\hat{L}_z\phi_{100}) + 16(\phi_{210}|\hat{L}_z\phi_{210}) + 8(\phi_{22-1}|\hat{L}_z\phi_{22-1})] \\
&= |C|^2[(0\hbar)(\phi_{100}|\phi_{100}) + 16(0\hbar)(\phi_{210}|\phi_{210}) + 8(-\hbar)(\phi_{22-1}|\phi_{22-1})] \\
&= -\frac{8\hbar}{25}.
\end{aligned}$$

(e) The probability of finding the electron in the $\varphi_{210}(\boldsymbol{r})$ state is just the square of the magnitude of its normalized coefficient which is $|C4i|^2 = 16/25 = 0.64$

(f) The eigenfunction, $\varphi_{100}(\mathbf{r})$, for a 1s electron of a hydrogen-like atom is given by,

$$\psi = ke^{-Zr/a_0}$$

where k is a constant, a_o is the radius of the first Bohr orbit (and for hydrogen Z =1). The radial probability density of finding the electron at distance r, irrespective of direction, is given by $P = 4\pi r^2\psi^2$, which can be written as $P = k_2 r^2 e^{-2Zr/a_o}$, where k_2 is a constant.

The radius at which P is maximum occurs when $\frac{dP}{dr} = 0$. We obtain the following:

$$\frac{dP}{dr} = k_2\left[2re^{-2Zr/a_0} - r^2 2\left(\frac{Z}{a_o}\right)e^{-2Zr/a_0}\right] = 0$$

$$\frac{dP}{dr} = k_2 2re^{-2Zr/a_0}\left(1 - \frac{Zr}{a_o}\right) = 0$$

$$\frac{Zr}{a_o} = 1$$

$$r = \frac{a_o}{Z}$$

(g) Each energy eigenstate evolves in time by a phase:

$$\psi_{nlm}(\vec{r}, t) = \psi_{nlm}(\vec{r}, 0)e^{-iE_n t/\hbar}.$$

Note that the energy (and therefore the time evolution) depends only on the principal quantum number n. Using the superposition principle, we have

$$\psi(\vec{r}, t) = C(\phi_{100}(\vec{r})e^{-iE_1 t/\hbar} + 4i\phi_{210}(\vec{r})e^{-iE_2 t/\hbar} - 2\sqrt{2}\phi_{22-1}(\vec{r})e^{-iE_2 t/\hbar}),$$

where $E_n = -13.6\,\text{eV}/n^2$ since we are dealing with hydrogen.