Elasticity

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When external forces are applied to a solid material, it may be reversibly deformed. Such behavior is the focus of elasticity theory. The forces may induce stress, tensile, shear, bending and torsion.

A. Stress

In order to introduce stress, let us begin with a problem. Suppose a small rock (say, $l = 2 \cdot 10^{-2}$ m in length) which is thrown towards a wall and bounces off it. What is the force, F, the rock experiences during the impact? (As usual, try estimating F yourself before reading on) What is the duration of the impact Δt?

From momentum conservation, the impulse of the an elastic impact is $2mv$ where m is the rock's mass and v is its initial and final velocity magnitude. From Newton's second law of motion, the characteristic force is $F \simeq 2mv/\Delta t$. The typical mass density of rock is a few times greater than the mass density of water, say $\rho \simeq 3$ ton m^{-3} so that $m \sim \rho l^3 \sim 24$ grams. A typical velocity is $v \sim 10$ m s^{-1} (say, one second of a free fall). The duration Δt should be about twice the time that a sound wave crosses the rock, $\Delta t \simeq 2l/c_s$, where c_s is the speed of sound in the rock's medium. The reason for the factor 2: a first sound wave starting from the wall side "informs" the other parts of the rock about the impact and when it arrives to the free side of the rock a second sound wave is returned and when it approach to the wall side, it causes the detachment of the rock from the wall. The only term we still have to estimate is c_s.

Instead of estimating the speed of sound, we use the deformation of the rock during the collision as motivation to discuss elasticity. For simplicity, consider a cylinder of length l and cross-sectional area S, compressed by $\delta l \ll l$ along its axis, as shown in Figure 1. The reactive force of the cylinder may be estimated by approximating it as a large spring, $F = -k\,\delta l$, which is composed of many small springs connected serially along l and in parallel along S. Therefore (in analogy to the conductance of connected resistors), $k \propto S/l$. The coefficient of proportionality is a property of the material, and is called Young's Modulus E.

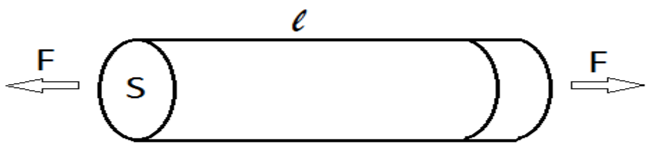


FIG. 1: Compressed cylinder.

Its units are $[E] = \mathrm{N\ m^{-2}}$, i.e. of pressure, and is typically of order 100 giga-pascal. Values for some materials are shown in Table I.

Material	E
Glass	50×10^9 Pa
Aluminum	70×10^9 Pa
Copper	100×10^9 Pa
Steel	200×10^9 Pa
Diamond	1200×10^9 Pa
Rubber	$(0.01\text{–}0.1) \times 10^9$ Pa

TABLE I: Young's moduli of several materials.

Therfore, the force is written as

$$F = -\left(\frac{ES}{l}\right)\delta l\,. \tag{1}$$

Notice that this is an extensive expression which can be written also in an intensive form,

$$\frac{F}{S} = -E\frac{\delta l}{l}\,, \tag{2}$$

where F/S is the stress σ, and $\delta l/l$ is the strain ϵ.

It may also be written as $\sigma_{xx} = -E\epsilon_{xx}$, which is a special case of the general expression $\sigma_{ij} = C_{ij;kl}\epsilon_{kl}$. Here, σ_{ij} is the (Cauchy) stress tensor, giving the normalized traction force across a suface via $T_j = dF_j/dS = \sigma_{ij}n_i$, where $\vec{T}$ and $\vec{F}$ are the stress and force vectors, and S and $\vec{n}$ give the area and normal of the surface of interest. The strain tensor is given by $\epsilon_{ij} = (\partial_j u_i + \partial_i u_j)/2$, where $\vec{u}$ is the material displacement.

It is useful to consider the energy change due to compression (or rarefaction). we use the analogy to springs again,

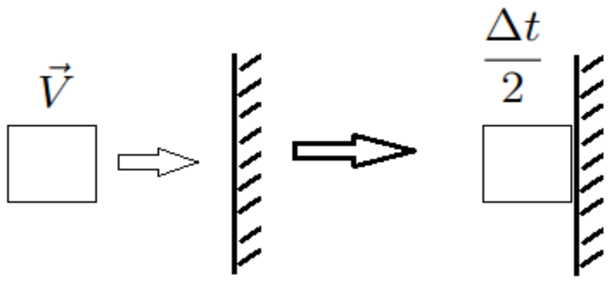
$$U = \frac{1}{2}k\,\delta l^2 = \frac{1}{2}ESl\left(\frac{\delta l}{l}\right)^2\,. \tag{3}$$

Since $V = Sl$ is the cylinder volume, the energy density is simply

$$\frac{U}{V} = \frac{1}{2}E\left(\frac{\delta l}{l}\right)^2 . \tag{4}$$

This is one component of the total strain energy density, $U/V = (1/2)\sigma_{ij}\epsilon_{ij}^2$.

Returning to the problem of the wall collision, it is possible divide the event into two parts: Impact and Rebound.



What is the deformation of the rock due to the collision?

$$\delta l = \frac{v}{2} \cdot \frac{\Delta t}{2}$$

The first factor of $\frac{1}{2}$ is due to a rough estimation of the average velocity of the rock which decelerates from v to zero. The Second factor of $\frac{1}{2}$ is due to an assumption that the impact is symmetric so that it takes the same time to the rock to change its velocity from v to 0 and from 0 to $-v$. From the impulse equation,

$$F = \frac{ES}{l}\delta l = \frac{2mv}{\Delta t},$$

$$\frac{ES}{l} \cdot \frac{v\Delta t}{4} = \frac{2mv}{\Delta t}.$$

We can substitute $m = S \cdot l \cdot \rho$ and get:

$$\frac{ES}{l} \cdot \Delta t = \frac{8}{\Delta t}\rho Sl \quad \Rightarrow \quad \Delta t \approx l \cdot 3\sqrt{\frac{\rho}{E}} = 3l\sqrt{\frac{\rho}{E}}$$

For example, a rock with $l = 2 \cdot 10^{-2}m$, $\rho = 3 \cdot 10^3 \frac{kg}{m^3}$, $E = 40 \cdot 10^9 Pa$ we get:

$$\Delta t = 3 \cdot 2 \cdot 10^{-2}\sqrt{\frac{3 \cdot 10^3}{40 \cdot 10^9}} \approx 3 \cdot 2 \cdot 10^{-2} \cdot 3 \cdot 10^{-4} \approx 10^{-5} = 10\mu s$$

The speed of sound is $c_s \simeq \sqrt{\frac{E}{\rho}}$ up to a numerical factor in the order of 1 which depends on the material properties. Therefore, we get that the final expression for Δt is close to the result that would be estimated by c_s considerations in the beginning of the chapter. In other words, using the theory of elasticity we have estimated the speed of sound of solids.

B. Shear

There are other ways to deform a material. Consider applying a force parallel to the surface, as in Figure 2. The relevant strain component is now $\varepsilon_{xy} = u/l$, and the stress-

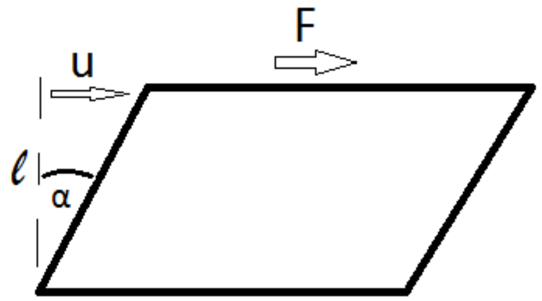


FIG. 2: Sheared material.

strain relation $\sigma_{xy} = C_{xyxy}\epsilon_{xy}$ is written as

$$\frac{F}{S} = -G\frac{u}{l}\,, \tag{5}$$

where G is the material's shear modulus. Similarly, the strain energy is given by

$$\frac{U}{V} = \frac{1}{2}G\left(\frac{u}{l}\right)^2 \tag{6}$$

Several examples:

Glass: $G = 25 \cdot 10^9 Pa$

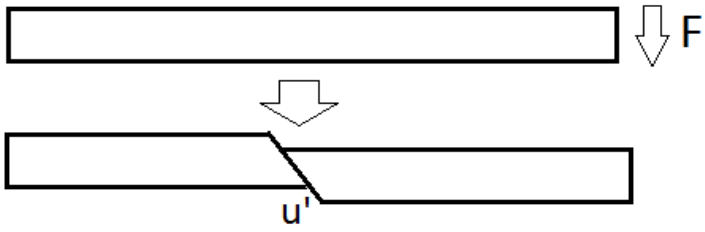
Copper: $G = 40 \cdot 10^9 Pa$

Steel: $G = 80 \cdot 10^9 Pa$

Applying the force in a different direction will create a Shear in the material.

In the following example, $u' = \sqrt{2}u$.

$$\frac{1}{2}G\left(\frac{u'}{l}\right)^2 = \frac{1}{2}E\left(\frac{u}{l}\right)^2 \quad \Rightarrow \quad E \approx 2G$$



C. Hearing

The range of Human hearing extends between $20Hz - 20,000Hz$. Our sense of hearing perceives equal ratios of frequencies as equal differences in pitch so that they are on a logarithmic scale. Hence, The middle is at about $\sqrt{20 \cdot 20,000} = 2\sqrt{10^5} \sim 600Hz$

Tuning forks allow musicians to hear a specific frequency in which they vibrate to create a single tone. For example, $A \to 440Hz, \quad C \to 262Hz$.

The rods on the fork are in essence oscillators that resonate in a specific frequency. Let l be the length of the rods at the top of the fork and S the surface of the top of the rods.

$$U = \frac{1}{2} G\alpha^2 \cdot S \cdot l$$

$$\tau = -\frac{\partial U}{\partial \alpha} = -GSl\alpha$$

Where τ is the torque.

$$\tau = I\ddot{\alpha}, \qquad I = \frac{Ml^2}{3}, \qquad M = \rho Sl$$

Where I is the moment of inertia.

$$\ddot{\alpha} + \frac{GSl}{I}\alpha = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{GSl \cdot 3}{\rho S l^3}} = \frac{1}{l}\sqrt{\frac{3G}{\rho}}$$

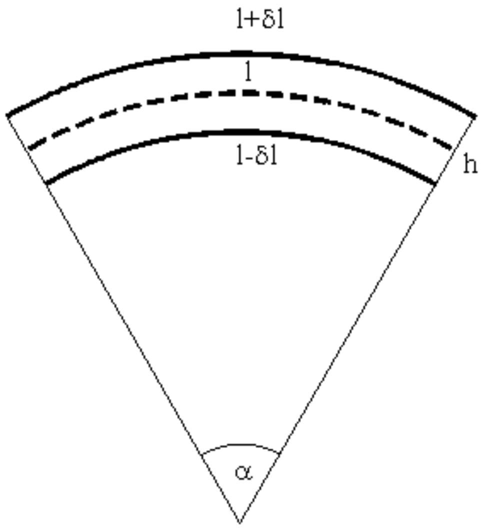
$$\nu = \frac{1}{2\pi}\omega = \frac{1}{2\pi l}\sqrt{\frac{3G}{\rho}}$$

In the class example, $l = 16cm, \quad G_{aluminum} = 25 \cdot 10^9 Pa$:

$$\nu = \frac{1}{2\pi \cdot 16 \cdot 10^{-2}} \cdot \sqrt{\frac{3 \cdot 25 \cdot 10^9}{3 \cdot 10^3}} = \frac{1}{2\pi \cdot 16 \cdot 10-2} \cdot 5 \cdot 10^3 \approx 5 \cdot 10^3 Hz$$

Which seems to be much higher tunned than the class example. Further, our frequency expression does not depend on the surface. Suspicious.

Let's take a look at the deformation of each rod:



$$l = \alpha R, \quad l + \delta l = \alpha(R + \frac{h}{2}) \quad \Rightarrow \quad \delta l = \frac{\alpha h}{2}$$

This deformation is a change of the bar's length - Young's Modulus.

$$U = \frac{1}{2}E\left(\frac{\delta l}{l}\right)^2\left(\frac{1}{2}\right)V = \frac{1}{16}E\left(\frac{h}{l}\right)^2 V\alpha^2 \tag{7}$$

The second factor of $\frac{1}{2}$ is because of the averaging.

Similar to the previous (wrong) solution,

$$I\ddot{\alpha} = \tau = -\frac{\partial U}{\partial \alpha} = -\frac{1}{8}E\left(\frac{h}{l}\right)^2 V\alpha.$$

We get:

$$\omega^2 = \frac{1}{8}E\left(\frac{h}{l}\right)^2\frac{V}{I} = \frac{1}{8}E\left(\frac{h}{l}\right)^2 \cdot \frac{3V}{\rho V l^2}$$

Where

$$I = \frac{Ml^2}{3} = \frac{\rho V l^2}{3}$$

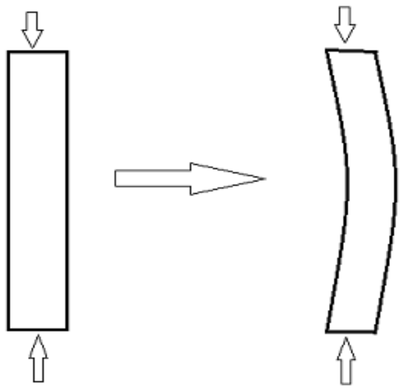
And the frequency in this case is

$$\nu = \frac{\omega}{2\pi} = \frac{1}{4\pi\sqrt{2}}\frac{h}{l^2}\sqrt{\frac{3E}{\rho}} = \frac{1.2}{4\pi} \cdot \frac{10^{-2}}{256 \cdot 10^{-4}} \cdot \sqrt{\frac{70 \cdot 10^9}{3 \cdot 10^3}} \simeq 10^{-2} \cdot 4 \cdot 5 \cdot 10^3 \approx 200Hz$$

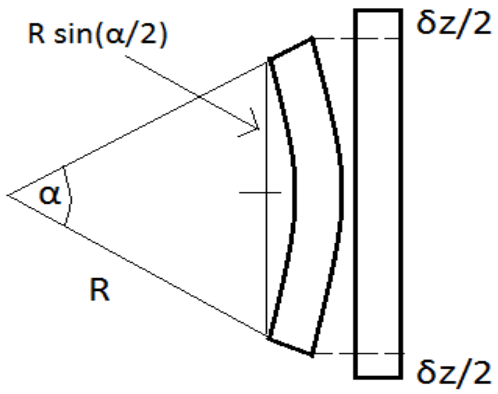
This results is much closer to the class example than the previous one. We can learn that the major effect in the tuning fork is the bending of its rods and not the shear oscillations of them.

D. Buckling Instability

1. Buckling energy and strain



When a force is applied from both ends of a rod, the system will tend to the lower energy state, which might be a buckling deformation. In this section we will estimate the conditions for a buckling transition.



$$\frac{U}{V} = \frac{1}{2}E\left(\frac{\delta l}{l}\right)^2, \quad \delta l = \alpha R - 2sin(\alpha/2)\cdot R$$

$$\epsilon = \frac{\delta l}{l} = \left(\frac{\alpha - 2sin(\alpha/2)}{\alpha R}\right)\cdot R = \frac{\alpha/2 - sin(\alpha/2)}{\alpha/2} \approx \frac{\alpha/2 - \alpha/2 + (\alpha/2)^3/6}{\alpha/2} = \frac{\alpha^2}{24}$$

Inserting this term in the expression of the energy from equation 7, we get

$$\frac{U_{buckling}}{V} = \frac{1}{16}E\left(\frac{h}{l}\right)^2\alpha^2 = \frac{3}{2}\cdot E\left(\frac{h}{l}\right)^2\cdot\frac{\delta l}{l} = \frac{1}{2}\cdot E\cdot\left(\frac{\delta l}{l}\right)^2 \tag{8}$$

$$\epsilon = \frac{\delta l}{l} = 3\left(\frac{h}{l}\right)^2 \tag{9}$$

The result is surprising, because it is not dependent on the rod's material.

2. *Transition to buckling*

Now that we have estimated expression for the energy and strain for a buckling state, let us compare it with the energy of the "compressed" state: The buckling energy grow linearly with the strain while the compressed state energy grows like the strain squared. Since the Force is the derivative of the energy, in the transition from compression to buckling (in which the energies of the two states are equal) there is a jump in the force which is decreased. It means that when applying a force on the object, in the beginning it's very hard to "squeeze" it, but after the critical point the item will "squeeze" immediately or break.
The critical strain is the buckling strain that we ha calculated, $\epsilon^* = \frac{3h^2}{l^2}$. The energy term which has to be minimized is $U' = U - F\delta z$
Before buckling (compression):

$$U = lS\frac{E}{2}\left(\frac{\delta l}{l}\right)^2 = lh^2\frac{E}{2}\left(\frac{\delta l}{l}\right)^2 \Rightarrow F_{\epsilon^*-0} = \frac{\partial U}{\partial \delta l} = \frac{Elh^2\delta l}{l^2} = 3Eh^2\frac{h^2}{l^2} \Rightarrow \sigma_{\epsilon^*-0} = \frac{F}{h^2} = 3E\frac{h^2}{l^2}$$

After buckling:

$$\sigma_{\epsilon^*+0} = \frac{F}{h^2} = \frac{3}{2}E\frac{h^2}{l^2}$$

When F increases, the slope of the line decreases and after critical point it becomes negative. After this point the minimum of energy is of the buckling state. An illustration of this transition is shown in Figure 3. In Figure 4 the displacement of z x vs. the stress is shown. Notice that after the transition to buckling it is much "easier" to "squeeze" the object.

E. Animal Bone Scaling

1. *Galilei*

Galilei has suggested the following simple model: the legs of the animal carry its mass, and there is a critical stress σ_{max}, in which its its leg bones break.

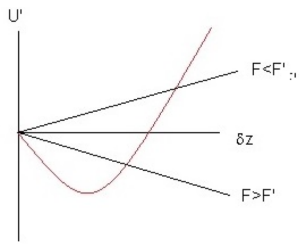


FIG. 3: energy before and after critical point

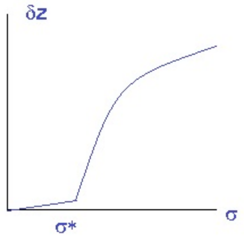


FIG. 4: δz as function of stress

M=mass of the animal, L= length of bone, h=thickness of bone

$$\sigma \sim \frac{Mg}{h^2} = \frac{L^3 \rho g}{h^2} = \beta \sigma_{max}$$

Where $\beta < 1$ is a factor that assures that the stress does not exceed its maximal value. Let us assume that this factor, β, animal's density ρ and the maximal stress σ_{max}, are the same for all of the animals (which are justified because they are made of similar materials). We get

$$\frac{L^3}{h^2} \sim const \Rightarrow L \sim h^{\frac{2}{3}}$$

This result has a smaller power than the real data.

2. Preventing Buckling

Let us assume that the major effect in the stress limitation is not bone breaking but bone buckling, so that $\sigma_{buckling} \leq \sigma_{max}$. Therefore,

$$\sigma_{buckling} = \frac{3}{2} E \frac{h^2}{L^2} \leq \sigma_{max} \Rightarrow \frac{h}{L} \sim \left(\frac{2\sigma_{max}}{3E} \right)^{\frac{1}{2}}.$$

Let us estimate the ratio between the height and the length of animals using:

$$\sigma_{max} \sim 200MPa, E = 20GPa$$

We get

$$\frac{h}{l} \sim \left(\frac{2 \cdot 2 \cdot 10^8}{3 \cdot 2 \cdot 10^{10}}\right)^{\frac{1}{2}} \sim 0.1$$

The result has a good agreement with the data only for small animals.

For big animals, the data shows $h \sim L^{0.74 \pm 0.08}$.

Instead of estimating the stress for bucking, let us use scaling considerations for this case (similar to Gelilei's solution):

$$\sigma \sim \frac{Mg}{h^2} = \frac{L^3 \rho g}{h^2} \sim \frac{3}{2} E \frac{h^2}{L^2} = \sigma_{buckling}$$

Assuming that the density and Young's modulus are the same for all of the animals, we get

$$L \sim h^{\frac{4}{5}}.$$

This result is closer but still not accurate.

3. Bending Stress

All of the estimations that we have made so far where under the assumption that the animals are "static". However, animals don't stand all of the time, but also walk and run. Let us model it by a force applied in the perpendicular direction to the bone and calculate the ratio between this force and bending.

$$\delta y = R\left[1 - Cos\left(\frac{\alpha}{2}\right)\right] \approx \frac{R}{2}\left(\frac{\alpha}{2}\right)^2 = \frac{L\alpha}{8}$$

Using the expressions for the energy (equation 8) and strain (equation 9) for buckling,

$$\frac{U}{V} = \frac{E}{4}\left(\frac{\alpha h}{2L}\right)^2 = 4E\left(\frac{h\delta y}{L^2}\right)^2$$

$$F = \frac{\partial U}{\partial \delta y} = 8ESL\frac{h^2}{L^4}\delta y \Rightarrow \delta y = \frac{FL^3}{8Eh^4}$$

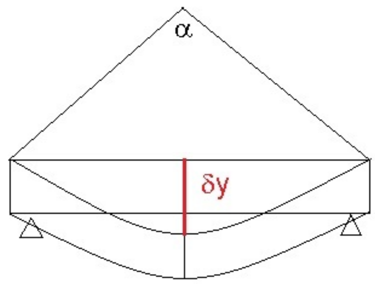


FIG. 5: bending

$$\sigma = \epsilon E = \frac{\alpha h E}{2L} = E\frac{4h\delta y}{L^2} = E\frac{4hFL^3}{L^2 8Eh^4} = \frac{FL}{2h^3} \sim \frac{MgL}{h^3} = \frac{L^4 g}{h^3} \Rightarrow L \sim h^{\frac{3}{4}}$$

This result is the closest to the data. The nature "wants" to keep σ below σ_{max}
Let us check it for elephants:
$M \sim 7000Kg, L \sim 1m, h \sim 0.1m \Rightarrow \sigma = \frac{7000\cdot 10\cdot 1}{4\cdot 2\cdot 0.1^3} \sim 1\cdot 10^7 Pa < \sigma_{max} \sim 200MPa$

The factor of 4 is added because elephants have 4 legs.

Lecture note 4: Fluid Dynamics and Waves

Lecture by Prof. Oleg Kiechevsky. Summary by dannyk, expanded and corrected by Eli Gudinetsky, July 2014

A. Fluid Dynamics

1. Viscosity

Consider a plate floating inside a large tank of liquid (for now, we will assume that the length of the plate L is much larger than the depth of the pool H, meaning $L >> H$) as illustrated in Figure 1. Moving the plate at a constant velocity v induces a velocity gradient

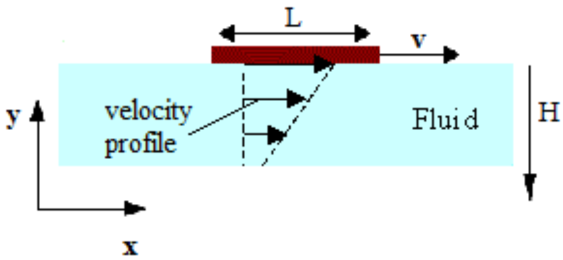


FIG. 1: A plate moving on top of a fluid.

in the liquid below. Each layer of liquid drags the layer below because of viscosity, so there is an downward flux of horizontal momentum. Each layer has an influx of momentum from the above layer and an outwards flux into the layer below. The force per unit area acting on the plate is therefore

$$\frac{F}{S} = \frac{\Delta p}{S\Delta t} = \eta \frac{\partial v_x}{\partial y} , \tag{1}$$

where F is the force applied on the plate for it to move at a constant velocity v, S is its area, $\Delta p/\Delta t$ is the change in momentum over time, v_x is the component of fluid velocity in the direction of the moving plate and η is the dynamic (shear) viscosity. For example, for water the dynamic viscosity is $\eta_{\mathrm{H_2O}} \approx 10^{-3}$ Pa $\cdot$ s.

The energy loss per unit volume is estimated as

$$\dot{\mathcal{E}} = -\frac{Fv}{Sy} = -\eta \left(\frac{\partial v}{\partial y}\right)^2 , \tag{2}$$

which is exact up to a factor of $1/2$. Here y denotes the coordinate along which the velocity of the fluid changes, so $\partial v_x/\partial y$ denotes the fluid's velocity gradient.

We can estimate how long it takes for a steady gradient to form. As $\Delta p/\Delta t \sim \eta S v/y$,

$$Ft \simeq mv \quad \longrightarrow \quad \eta \frac{v}{y} St \sim \underbrace{\rho S y}_{m} \underbrace{\frac{v}{2}}_{\text{avg. velocity}} , \tag{3}$$

and therefore $t \sim y^2\rho/2\eta$. This means that after a time t, the gradient reaches down to a depth

$$y \sim \sqrt{2\eta t/\rho} = \sqrt{2\nu t}\,, \tag{4}$$

where $\nu = \eta/\rho$ is the kinematic viscosity.

Notice that for gases, the kinematic viscosity is equal to the diffusion coefficient, $\nu = D = \lambda\bar{v}/3$, where λ is the mean free path and $\bar{v}$ is the thermal velocity. We can estimate ν using $\bar{v} \sim \sqrt{3k_BT/m} \sim 500$ m/s and $\lambda = 1/(\pi d^2 n)$, where d is the molecule diameter and n is the number density of the gas. For air, for example, $\nu_{\text{air}} \sim 1.6 \times 10^{-5}$ m^2/s.

Let us try to estimate the flow velocity of an inclined river. Let us denote by $\alpha \ll 1$ the inclination angle, H the depth of the water, and v_m the river's maximal velocity, as seen in Figure 2.

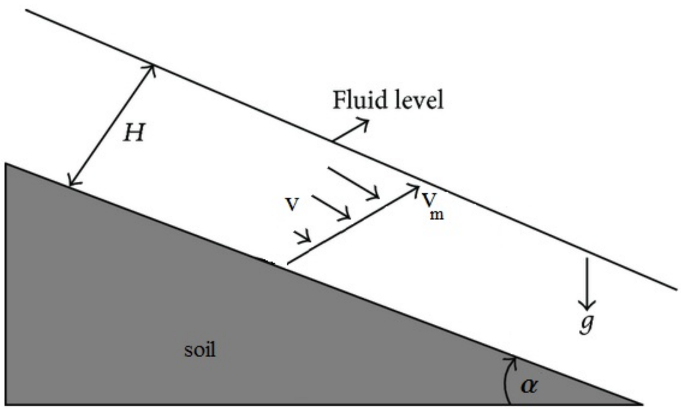


FIG. 2: A cross section of a river

The energy loss is due to viscosity and is balanced by the gravitational energy,

$$\dot{U} = mg\dot{h} = mgv\alpha = mg\frac{v_m}{2}\alpha \overset{!}{=} \eta\left(\frac{v_m}{H}\right)^2 SH \sim \nu m \left(\frac{v_m}{H}\right)^2 , \tag{5}$$

where S is some unit area of the fluid. This implies that $v \sim v_m/2 \sim gH^2\alpha/4\nu$. Estimating v for a puddle on an inclined road with $H \sim 1$ mm, $\alpha \sim (10\text{ cm})/(10\text{ m}) \sim 10^{-2}$, gives $v \sim 2$ cm/s, which is reasonable. Estimating v for a river with $H \sim 10$ m and $\alpha \sim$

(0.3 m)/(3000 m) $\sim 10^{-4}$ gives $v \sim 2 \times 10^4$ m/s, which is clearly wrong. Can you guess what we did wrong? We will discuss this later on.

2. Drag force

Let us consider a sphere of radius R moving through a fluid with velocity u, e.g. a rain drop falling through air. After a while the a velocity gradient is formed and it is stationary in the sphere frame. Consider spherical shells at a distance r from our sphere. In a steady state, the flux of fluid momentum crossing each shell is constant. This means that $\Delta p/\Delta t = S\eta\nabla v =$ const., where S is the area of a spherical shell of radius r, and v is the velocity of the fluid. Writing $S \propto r^2$ and treating the flow as spherically symmetric (although it does not really have such symmetry) gives

$$\frac{\partial v}{\partial r} r^2 = \text{const} \quad \longrightarrow \quad v \sim \frac{1}{r} \,. \tag{6}$$

Assuming the sphere with radius R moves with velocity u, we can solve the above equation with the boundary condition of no flow perpendicular to the sphere, and get $v \sim uR/r$. Again, the force per unit area is determined by the velocity gradient, which means:

$$\frac{F_{\text{drag}}}{4\pi R^2} = \eta \frac{u}{R} \quad \longrightarrow \quad F_{\text{drag}} = 4\pi\eta R u \,, \tag{7}$$

this approximation reproduces Stokes' formula, $F_{\text{drag}} = 6\pi\eta R u$.

Lets estimate the velocity of a rain drop using Stokes' formula. There is a force balance between gravity and drag which gives

$$mg = F_{\text{drag}} \quad \longrightarrow \quad u \sim \frac{mg}{6\pi\eta R} \,. \tag{8}$$

Estimating the diameter of a drop to be $d = 3$ mm we get $v \sim 200$ m/s, which is, again, clearly wrong. The following section will give an explanation about the origin of our error.

3. Skin-layer

Looking back at our first problem of a plate in a pool we notice, as shown in Eq. (4), that the velocity profile should reach the maximum depth of $y_{\text{max}} = \sqrt{2\nu t_{\text{max}}} \sim \sqrt{2\nu L/u}$, where L is the plates length. This length scale is much smaller than the pool's depth, $L \ll H$.