Lecture note 4: Fluid Dynamics

Lecture by Prof. Oleg Kiechevsky. Summary by dannyk, expanded and corrected by Eli Gudinetsky, July 2014

A. Fluid Dynamics

1. Viscosity

Consider a plate floating inside a large tank of liquid (for now, we will assume that the length of the plate L is much smaller than the depth of the pool H, meaning $L \ll H$) as illustrated in Figure 1. Moving the plate at a constant velocity v induces a velocity gradient

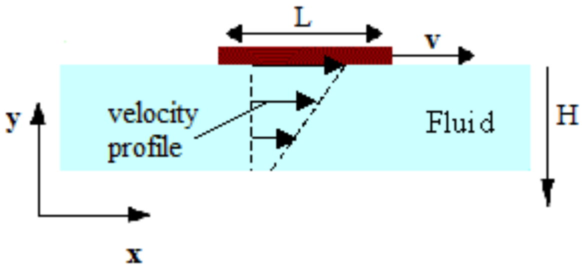

FIG. 1: A plate moving on top of a fluid.

in the liquid below. Each layer of liquid drags the layer below because of viscosity, so there is an downward flux of horizontal momentum. Each layer has an influx of momentum from the above layer and an outwards flux into the layer below. The force per unit area acting on the plate is therefore

$$\frac{F}{S} = \frac{\Delta p}{S\Delta t} = \eta \frac{\partial v_x}{\partial y} \,, \tag{1}$$

where F is the force applied on the plate for it to move at a constant velocity v, S is its area, $\Delta p/\Delta t$ is the change in momentum over time, v_x is the component of fluid velocity in the direction of the moving plate and η is the dynamic (shear) viscosity. For example, for water the dynamic viscosity is $\eta_{\mathrm{H_2O}} \approx 10^{-3}$ Pa · s.

The energy loss per unit volume is estimated as

$$\dot{\mathcal{E}} = -\frac{Fv}{Sy} = -\eta \left(\frac{\partial v}{\partial y}\right)^2 \,, \tag{2}$$

which is exact up to a factor of 1/2. Here y denotes the coordinate along which the velocity of the fluid changes, so $\partial v_x/\partial y$ denotes the fluid's velocity gradient.

We can estimate how long it takes for a steady gradient to form. As $\Delta p/\Delta t \sim \eta S v/y$,

$$Ft \simeq mv \quad \longrightarrow \quad \eta \frac{v}{y} S t \sim \underbrace{\rho S y}_{m} \underbrace{\frac{v}{2}}_{\text{avg. velocity}} , \tag{3}$$

and therefore $t \sim y^2 \rho / 2\eta$. This means that after a time t, the gradient reaches down to a depth

$$y \sim \sqrt{2\eta t/\rho} = \sqrt{2\nu t} \,, \tag{4}$$

where $\nu = \eta/\rho$ is the kinematic viscosity.

Notice that for gases, the kinematic viscosity is equal to the diffusion coefficient, $\nu = D = \lambda \bar{v}/3$, where λ is the mean free path and $\bar{v}$ is the thermal velocity. We can estimate ν using $\bar{v} \sim \sqrt{3k_B T/m} \sim 500$ m/s and $\lambda = 1/(\pi d^2 n)$, where d is the molecule diameter and n is the number density of the gas. For air, for example, $\nu_{\text{air}} \sim 1.6 \times 10^{-5}$ m^2/s.

Let us try to estimate the flow velocity of an inclined river. Let us denote by $\alpha \ll 1$ the inclination angle, H the depth of the water, and v_m the river's maximal velocity, as seen in Figure 2.

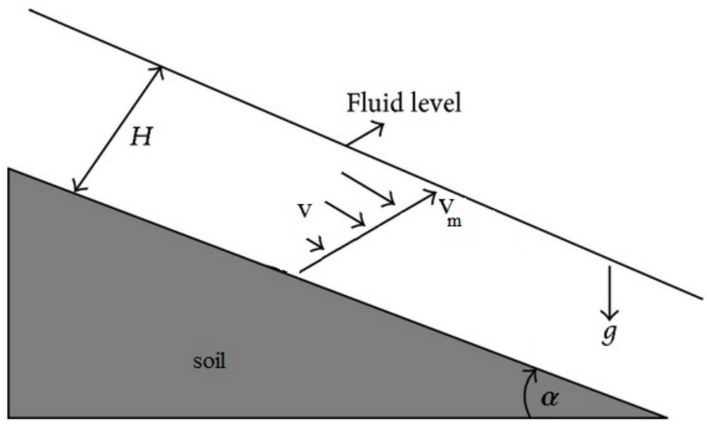

FIG. 2: A cross section of a river

The energy loss is due to viscosity and is balanced by the gravitational energy,

$$\dot{U} = mg\dot{h} = mgv\alpha = mg\frac{v_m}{2}\alpha \overset{!}{=} \eta \left(\frac{v_m}{H}\right)^2 SH \sim \nu m \left(\frac{v_m}{H}\right)^2 , \tag{5}$$

where S is some unit area of the fluid. This implies that $v \sim v_m/2 \sim gH^2\alpha/4\nu$. Estimating v for a puddle on an inclined road with $H \sim 1$ mm, $\alpha \sim (10\text{ cm})/(10\text{ m}) \sim 10^{-2}$, gives $v \sim 2$ cm/s, which is reasonable. Estimating v for a river with $H \sim 10$ m and $\alpha \sim$

(0.3 m)/(3000 m) $\sim 10^{-4}$ gives $v \sim 2 \times 10^4$ m/s, which is clearly wrong. Can you guess what we did wrong? We will discuss this later on.

2. Drag force

Let us consider a sphere of radius R moving through a fluid with velocity u, e.g. a rain drop falling through air. After a while the a velocity gradient is formed and it is stationary in the sphere frame. Consider spherical shells at a distance r from our sphere. In a steady state, the flux of fluid momentum crossing each shell is constant. This means that $\Delta p/\Delta t = S\eta\nabla v = \text{const.}$, where S is the area of a spherical shell of radius r, and v is the velocity of the fluid. Writing $S \propto r^2$ and treating the flow as spherically symmetric (although it does not really have such symmetry) gives

$$\frac{\partial v}{\partial r} r^2 = \text{const} \quad \longrightarrow \quad v \sim \frac{1}{r}\,. \tag{6}$$

Assuming the sphere with radius R moves with velocity u, we can solve the above equation with the boundary condition of no flow perpendicular to the sphere, and get $v \sim uR/r$. Again, the force per unit area is determined by the velocity gradient, which means:

$$\frac{F_{\text{drag}}}{4\pi R^2} = \eta \frac{u}{R} \quad \longrightarrow \quad F_{\text{drag}} = 4\pi\eta R u\,, \tag{7}$$

this approximation reproduces Stokes' formula, $F_{\text{drag}} = 6\pi\eta R u$.

Lets estimate the velocity of a rain drop using Stokes' formula. There is a force balance between gravity and drag which gives

$$mg = F_{\text{drag}} \quad \longrightarrow \quad u \sim \frac{mg}{6\pi\eta R}\,. \tag{8}$$

Estimating the diameter of a drop to be $d = 3$ mm we get $v \sim 200$ m/s, which is, again, clearly wrong. The following section will give an explanation about the origin of our error.

3. Skin-layer

Looking back at our first problem of a plate in a pool we notice, as shown in Eq. (4), that the velocity profile should reach the maximum depth of $y_{\text{max}} = \sqrt{2\nu t_{\text{max}}} \sim \sqrt{2\nu L/u}$, where L is the plates length. This length scale is much smaller than the pool's depth, $L \ll H$.

Hence, the characteristic length is not the pool depth but rather the maximum depth of the profile, or the 'skin-layer'.

In the case of a spherically symmetric drop with diameter d, the same argument indicates a skin-layer of roughly $\delta \sim \sqrt{2\nu d/u}$. In the case where $\delta \gg d$, there is no skin-layer effect and the information about the moving sphere spreads to infinity. In the case where $\delta \ll d$, the velocity gradient is not u/R but rather u/δ.

Let us calculate the ratio δ/d:

$$\frac{\delta}{d} = \sqrt{2\nu\frac{1}{ud}} = \sqrt{\frac{2}{\mathrm{Re}}} \sim \mathrm{Re}^{-1/2}\,, \tag{9}$$

where Re is Reynolds number. This is useful since we can decide whether skin-layer effects are important or not merely by estimating Re.

In the case of a rain drop, $\mathrm{Re} \sim 200 \times 3 \times 10^{-3}/1.6 \times 10^{-4} \sim 2 \times 10^4 \gg 1$, which means that in order to get a correct free fall velocity we must consider the skin-layer. Correcting our calculations by taking the gradient to be u/δ, we get

$$F_{\mathrm{drag}} = 4\pi R^2 \eta \frac{u}{\delta} \simeq \pi d \eta u \sqrt{\frac{ud}{\nu}}\,. \tag{10}$$

By comparing gravity to drag as above, it follows that the terminal velocity of the drop is $u \sim 15$ m/s, which is much more reasonable than our earlier estimate.

Another way of viewing drag is by inspecting the force a body must exert on the surrounding fluid particles it encounters on its way. The momentum that a spherically shaped droplet transfers to the environment is

$$\Delta p = \underbrace{Su\Delta t}_{\text{Volume covered in time } \Delta t} \rho u\,, \tag{11}$$

so the force is

$$F = \frac{\Delta p}{\Delta t} = S\rho u^2 \propto u^2\,, \tag{12}$$

proportional to u^2. However, this is not true in ideal fluids, for which the same momentum the sphere transfers forward, it gains from the back. We can understand this by approximating the motion of a sphere as it being composed of a series of spheres, each inflating as the previous sphere deflates slightly behind it. The momentum flux is conserved over the area of a sphere, so $v(r) \propto 1/r^2$, like the electric field of a point charge. These velocity fields are analogous to an electric dipole, which is symmetric along its axis. `[There are quite a few spelling errors. Please run a spell checker.]`

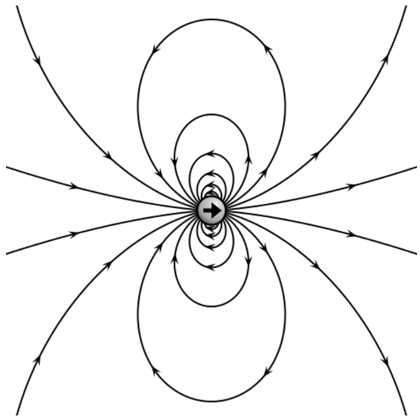

FIG. 3: The velocity field generated by a moving sphere

In this case, the net momentum is constant: the momentum given to the fluid is transferred to the back of the sphere, hence $\vec{F} = 0$. However, experimental results show that for large Reynolds numbers, the force actually is proportional to u^2. We'll try to understand this in the following section.

4. Navier-Stokes Equation

The equation we'll provide (without proof) is actually Newton's second law for an element of fluid which moves with velocity $\vec{v}$,

$$\rho \frac{d\vec{v}}{dt} = -\nabla P + \eta \nabla^2 \vec{v} + \rho \vec{g} \tag{13}$$

Some intuition: if there is no pressure gradient, there would be no momentum change, so we have the first derivative of P. The viscosity term involves a second derivative of $\vec{v}$ because the fluid element experiences two forces (each of them proportional to the first derivative of velocity) in opposite directions - one from each side of the fluid element. The last term gives the gravitational acceleration, $d\vec{v}/dt = \vec{g}$.

Eventually we would like to obtain an expression for $\vec{v}(\vec{r}, t)$. The velocity of the moving element of fluid changes both with time and with position,

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial \vec{r}} d\vec{r} \tag{14}$$

and

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \frac{\partial \vec{v}}{\partial \vec{r}}, \tag{15}$$

so using

$$\vec{v}\frac{\partial \vec{v}}{\partial \vec{r}} = v_i \frac{\partial v_j}{\partial x_i} = (\vec{v}\cdot\nabla)\vec{v}\,, \tag{16}$$

we conclude that

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho} + \nu\nabla^2\vec{v} + \vec{g}\,. \tag{17}$$

This equation has numerous consequences. For example, we can deduce that winds must exist. Indeed, consider a planar atmosphere, with gradients only in the z direction. If we assume that $\vec{v} = 0$ (mechanical equilibrium), we obtain hydrostatic equation $\nabla P = \rho\vec{g}$, where for an ideal gas $P = RT\rho/M$. Here R is the ideal gas constant, T is the temperature, ρ is the density and M is the fluid's molar mass. From the hydrostatic equation it follows that $P = P(z)$ and $\rho = \rho(z)$, so $T = T(z)$. This means that every point with the same altitude in the atmosphere has the same temperature, which we know isn't true. This means that the atmosphere must not be static, i.e. winds must exist.

The Navier-Stokes equation is non-linear. In such equations, the solutions are usually unstable. For a small change in the initial conditions, the solutions will diverge from each other by the Lyapunov exponent, $e^{\lambda t}$, where λ^{-1} is the characteristic time in the problem. For winds on earth, the characteristic time will be

$$T \sim \frac{R_\oplus}{v_{\text{wind}}} \sim \frac{10^7 \text{ m}}{10 \text{ m/s}} \approx 10 \text{ days}\,. \tag{18}$$

This time is the upper limit for a weather forecast - For larger times, small perturbations make it impossible to solve the equations.

5. The Reynolds Number

We can now define the Reynolds number formally using the Navier-Stokes equation, as the ratio between the nonlinear inertial term and the viscous term,

$$Re \equiv \frac{(\vec{v}\cdot\nabla)\vec{v}}{\nu\nabla^2\vec{v}} \sim \frac{\rho v v/L}{\eta v/L^2} = \frac{\rho v L}{\eta} = \frac{vL}{\eta}\,, \tag{19}$$

where L is some characteristic length scale, v is some characteristic velocity, and $\rho\,,\nu$ and η are the fluid density, dynamic viscosity and kinematic viscosity, respectively.

For $Re \gg 1$, we can neglect the viscosity and treat the problem as flow in an ideal fluid. In such a fluid, unlike our previous description, the flow is not forward-backward symmetric, and the sphere doesn't acquire momentum from its back. Instead, turbulence forms behind it, and the momentum which is transferred from the forward direction is directed to the turbulent region.

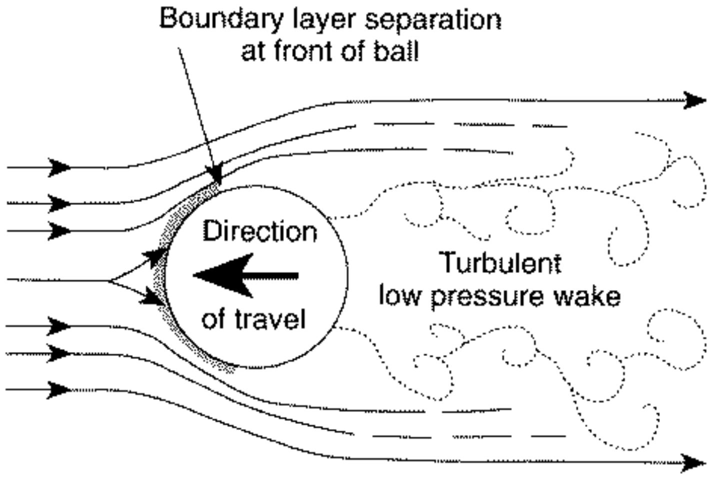

FIG. 4: Illustration of a flow past a sphere [Ref]

Using the same considerations as above, we conclude that the drag force in the $Re \gg 1$ regime is $F_D = \rho S u^2$, so the terminal velocity of a droplet is $u \sim 5$ m/s, which is approximately the correct result.

Airplanes use the same principle to fly - turbulence forms behind the lower edge of the wing, and momentum is transferred to the air molecules in this regime. As a result, the plane acquires momentum in the opposite direction - up into the air.

Let's calculate the ratio between the drag force we've found and the Stokes force,

$$\frac{F_D}{F_S} = \frac{\rho S u^2}{6\pi\eta R u} = \frac{Ru}{6\nu} \sim Re\,. \tag{20}$$

We may define an effective kinematic viscosity, $\nu^* = Ru/6$, such that for $\nu = \nu^*$ these forces are equal, $F_D = F_S$.

We note that in gases, the kinematic viscosity, $\nu = \lambda\bar{v}/3$, is the same as the effective kinematic viscosity ν^* up to a numerical factor. The characteristic length and velocity scales are the mean free path and the thermal velocity.

Now we return to the flowing river problem. We use the same expression for velocity as before, but replace ν by the effective kinematic viscosity,

$$v = \frac{\alpha g H^2}{4\nu^*}\,,\ \nu^* = \frac{1}{6}Hv \quad \longrightarrow \quad v = \sqrt{\alpha g H} = \sqrt{10^{-4}\cdot 10\cdot 10} = 0.1 \text{ m/s}\,, \tag{21}$$

which now is reasonable.