Lecture 12: Standard Model

Prepared by Arnon Ben-Meir (revised by Uri), July 10, 2013

A. Introduction

In this (fourth) part of the course we proceed beyond the atom, down to the structure of the nucleus and the fundamental constituents of Nature.

The key insight we shall need is that the fundamental forces and processes are best described as interactions between real and virtual particles. We begin by discussing the standard model of particle physics.

Some references for this section include

- Quantum Field theory in a Nutshell (Zee); An introduction To Quantum Field Theory (Peskin & Schroeder).
- Elementary Particle Physics (Kenyon); Introduction to Elementary Particles (Griffiths).
- The Los Alamos Primer (Serber); An Introduction to Nuclear Physics (Greenwood & Cottingham).
- Review papers, e.g. gr-qc/9512024, astro-ph/030307, ...

B. Overview of the Standard Model

The Standard Model (SM) of particle physics is a theory concerning the fundamental forces in nature. It includes the electromagnetic, weak, and strong nuclear interactions, which mediate the dynamics of the known subatomic particles.

Figure 1 illustrates the fundamental particles of the standard model and their interactions. The left side of the figure shows fermions, while the right side shows bosons. Each fermion has an anti-particle, represented by a bar, e.g. the anti-particle of an electron neutrino is denoted $\bar{\nu}_e$. The anti-particles of the bosons are the same or similar SM bosons: the H and Z are their own anti-particles, the W^+ anti particle is W^- and vice versa, and the anti-particle of a gluon (say, a red-anti blue gluon) is yet another gluon (a blue-anti red gluon in this case).

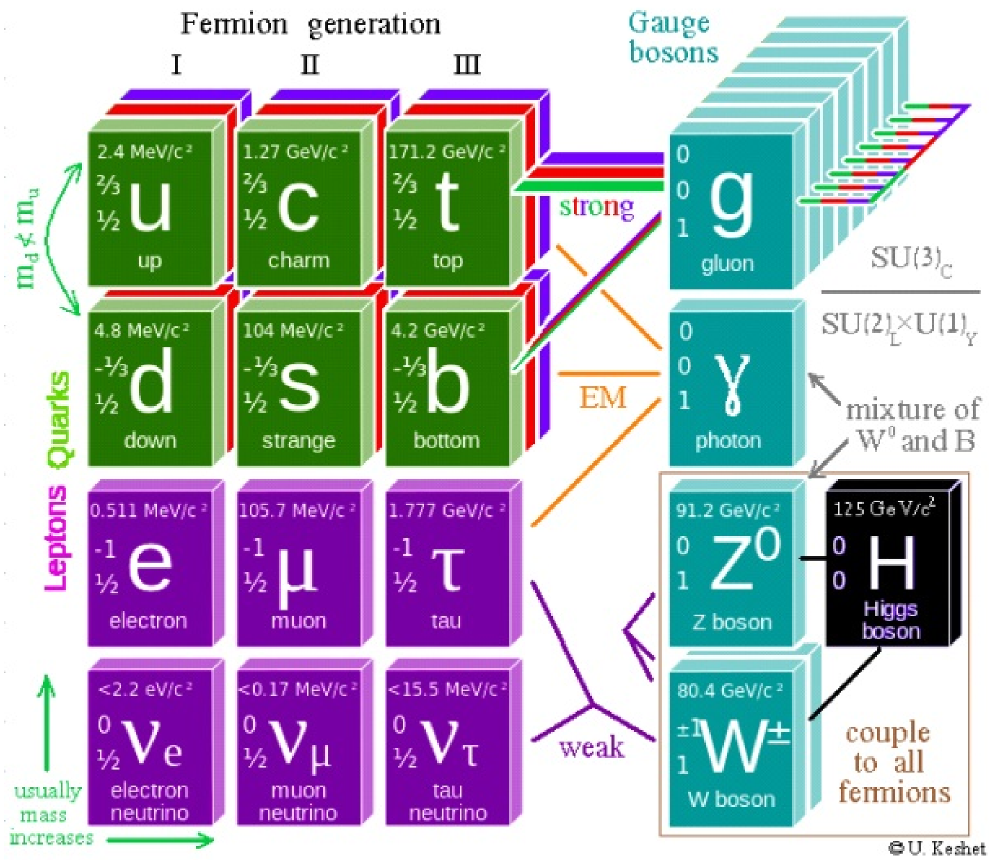

FIG. 1: A sketch of the Standard Model, illustrating the elementary particles and their interactions. Each particle cube shows the particle's name, mass, electric charge, and spin. Particle masses increase to the right and upward, except for the massless bosons and the $u-d$ anomaly.

1. SM Fermions

The SM fermions carry spin $s = 1/2$ and follow Pauli's exclusion principle. The SM recognizes two types of elementary fermions: quarks and leptons. The main difference is that quarks undergo strong interactions whereas the leptons do not. Particles that feel the strong force carry a strong charge known as color charge. Such a particle can be thought of as having one of three colors, arbitrarily labeled as blue, green, and red, or one of the three anti-colors. This renders quarks the fundamental constituents of matter. They combine to form composite, color-neutral (i.e., color singlet) particles called hadrons: either color triplets qqq known as baryons (like protons and neutrons) or color-anti color doublets $q\bar{q}$ known as mesons (like pions). For example, the lightest (and therefore most stable) baryon is the proton, which is made of 3 quarks: uud. The second lightest is the neutron, udd, which is slightly heavier than the proton because $m_d > m_u$. The lightest mesons are the pions, namely the $\pi^+ : u\bar{d}$ of electric charge $2/3 - (-1/3) = +1$, the $\pi^- : d\,\bar{u}$ of electric charge $-1/3 - 2/3 = -1$, and the electrically neutral $\pi^0 : u\bar{u} - d\bar{d}$.

Most of the physics we deal with can be understood in terms of the first generation of SM fermions: the u and d quarks, and the e and ν_e leptons. A physicist should know at least this generation by heart - *memorize it!* It turns out that there are three generations of fermions in the SM. The reason for this — and for much of the structure of the SM — remains unknown.

2. SM bosons

The right hand side of Figure 1 shows the SM bosons, which are (with the exception of the Higgs) known as gauge bosons. These bosons are an inevitable part of the gauge symmetries (or redundancies, depending your point of view) underlying the SM. The idea is that physics (formally: the Lagrangian) is symmetric under some local operation (formally: a phase change, a la $\psi \to e^{i\phi(x)}\psi$ for the wavefunction), but some messenger (namely, a massless gauge boson) is needed in order to convey the phase information between spatially distant points and guarantee the symmetry.

We know of four fundamental interactions: gravity, electromagntic, strong, and weak. Gravity is thought to be mediated by the graviton, denoted $\mathcal{G}$ — a massless boson of spin 2 — but this is not part of the SM and we defer it to later in the course. Each of the other three interactions can be understood as arising from a certain gauge symmetry, and is mediated by its own spin 1 gauge boson. The photon, denoted γ, mediates the electromagnetic interaction, and couples to all particles carrying an electric charge. The W^+, W^- and Z^0 mediate the weak interaction, and couple to all particles carrying a weak isospin: this includes all left-handed [1] fermion doublets such as $(ud)_L$ or $(e\nu_e)_L$. (An exception is the Z^0, which weakly couples to right-handed particles as well). The eight gluons, denoted G,

[1] Handedness refers to the helicity of the particle, namely if its spin is aligned towards the direction of motion (right-handed) or opposite to it (left handed). The relevant operator is $h \equiv \vec{J} \cdot \hat{p} = (\vec{L} + \vec{S}) \cdot \hat{p} = \vec{S} \cdot \hat{p}$. For massless particles, this is equivalent (up to $\hbar/2$) to the chirality. For massive particles, chirality refers to an internal property of the particle, which we shall not have time to discuss.

mediate the strong interaction, and couple to all particles carrying a color charge: this is the force holding hadrons together, and its residual effect holds nucleons together in a nucleus in spite of the electric repulsion between protons.

An interesting complication is that, as the W and Z bosons and the G gluons carry their respective charges, the W and Z^0 couple to each other, and the G couple among themselves. The nature and interactions of the gauge bosons are dictated by — and were originally understood from — the group structure of the corresponding inner symmetries, as mentioned briefly above. The strong and weak symmetries are non-abelian, hence the interaction between their force-carrying bosons.

3. Symmetry structure - rough idea

The electromagnetic interaction, given by the theory known as quantum electrodynamics (QED), is generated by a $U(1)$ symmetry denoted $U(1)_E$. What this means, is that if one assumes a local symmetry in which physics remains invariant under an arbitrary and non-homogeneous change to the phase of the wavefunction, $\psi(x) \to \psi(x)e^{\phi(x)}$, then one is directly and inevitably led to Maxwell's equations! (This beautiful derivation is unfortunately beyond the scope of this course.) The strong interaction corresponds to an $SU(3)$ symmetry, denoted $SU(3)_C$ for color, giving rise to the theory of quantum chromodynamics (QCD). The weak interaction corresponds to an $SU(2)$ symmetry, denoted $SU(2)_L$ for left-handed interactions, giving rise — together with the electromagnetic force — to the electroweak theory. The total group structure of the SM is therefore

$$SU(3)_C \times SU(2)_L \times U(1)_Y \,, \tag{1}$$

where subscript Y (for hypercharge) replaces E to denote the electroweak unification between electromagnetic and weak interactions at high energy (we will estimate this later). The theory generated by such symmetry groups (in general, $SU(N)$ or any compact group generated by a direct sum of Lie algebras, but don't worry if you don't know what all this means) is called a Yang-Mills theory (YM). We will not discuss the SM symmetry structure any further, as this would become too formal.

We will also not discuss the Higgs mechanism in any detail. The idea here is that all SM particles — including the electron, proton, and any other favorite particle of yours — are in fact massless! Their perceived masses arise entirely from their interaction (as energy$\sim$mass) with a charge-neutral spin-0 boson known as the Higgs, discovered only recently, in 2012 by the LHC.

Q: Does the massless character of all SM particles surprise you? It should not! What is the natural scale for a fundamental particle mass?

A: We already saw that the fundamental scale in Nature, given by the three fundamental constants $\{G, \hbar, c\}$, is the Planck scale, so a natural guess for the mass of a fundamental particle would be $\sim 10^{19}$ GeV! We cannot create such massive particles in our low energy Universe, so it is believed that all the particles we see — the entire SM — is only the massless sector of the true theory!

4. Plan

What we *will* do is review the SM at the back of the envelope level, avoiding the formalism and lengthy computations typically needed in a rigorous course. We begin by investigating the relation between a force and the boson mediating it. For example, we will study:

- How does the radial behavior of a force depend on the force carrier. For example, why are the electromagnetic force and gravity proportional to $1/r^2$?
- When is the interaction between like particles attractive, and when is it repulsive?
- Why are the above four forces the only ones we know of? Can other elementary forces exist?

Before we embark on this task, a few reminders are in order.

C. Reminder: electromagnetic scales

We have discussed three length scales associated with the electron: the classical radius r_e, the (reduced) Compton length $\lambda \equiv \lambda_c/(2\pi)$, and the Bohr radius a_B.

The classical radius of the electron is the size an electron would need to have have such that its mass can be entirely attributed to electrostatic potential energy: $r_e = e^2/(m_e c^2) \sim 3$ f (recall that 1 f $= 10^{-13}$ cm). This is the typical scale in which a free electron responds to an electromagnetic wave; indeed, $\sigma_T = (8\pi/3) r_e^2$.

The Compton wavelength, $\lambda = \lambda_c/(2\pi) = \hbar/(m_e c) \sim 400$ f, is of central importance to our discussion. Do you remember what it signifies? We will come back to this in a bit.

The Bohr radius, $a_B = \hbar^2/(e^2 m_e) \simeq 0.5\mathring{A}$, is the characteristic size of an atom – any atom: the outermost electron feels an electrostatic force from the Z protons in the nucleus, but (roughly speaking) $Z-1$ of these protons are screened by the inner electrons. Hence, the force is hydrogen-like, and this electron's wavefunction is mostly localized near a_B.

These three length scales are separated by the electromagnetic fine structure constant, α_E. We henceforth retain the subscript E in order to distinguish it from the coupling constants of the other forces, α_G, α_S, and α_W. From smallest to largest, these scales may be written as

$$r_e \xrightarrow[/\alpha_E]{} \lambda \xrightarrow[/\alpha_E]{} a_B\,. \tag{2}$$

The elementary particle bound by the electromagnetic interaction is the atom, so we review some associated scales. The above argument implies that the typical energy of the outermost electron in the atom may be found, as in the hydrogen atom, from the uncertainty principle, giving $E \simeq 1\text{Ry} = (1/2)m_e(\alpha_E c)^2$. As the kinetic energy $(1/2)mv^2$ is of the order of E, the typical velocity of this electron (normalized by the speed of light) is $\beta \equiv v/c \simeq \alpha_E$. Inner electrons are bound at smaller radii, and consequently have larger energies, and correspondingly higher velocities, so generally speaking $\beta \gtrsim \alpha_E$. For the innermost orbital, the uncertainty principle argument yields $\beta \simeq Z\alpha$, which for high atomic numbers can approach a fair fraction of the speed of light. For example, in U^{92} the binding energy of the innermost electron is ~ 116 keV, corresponding to $\beta \simeq 92\alpha \simeq 0.6$.

Q: Which of the above length scales is typically associated with the transition between classical and quantum behavior?

A: This is, as usual, the deBrogli length,

$$\lambda_{dB} = \frac{\hbar}{p} = \frac{\hbar}{\gamma\beta mc} = \frac{\lambda}{\gamma\beta}\,, \tag{3}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. In the atom this becomes $\lambda_{dB} \lesssim \lambda/\beta \lesssim \lambda/\alpha_E = a_B$, so electron orbitals in the atom are just inside the quantum regime.

Note that $\gamma\beta$, appearing in Eq. (3), is a useful quantity when you're not sure if velocities are relativistic or not, because it is approximately

$$\gamma\beta = \sqrt{\gamma^2 - 1} \simeq \begin{cases} \beta & \text{if } v \ll c\ ; \\ \gamma & \text{if } \gamma \gg 1. \end{cases} \tag{4}$$

D. The Compton wavelength and QFTs

So, what is the significance of λ? One of the roles played by λ is the lengthscale below which one must resort to quantum field theory (QFT). The reason is that on scales $< \lambda$, the number of particles is no longer conserved: unavoidable interactions with quantum fluctuations can lead to the creation or annihilation of like particles. To keep track of all these particles, it is convenient to group them into fields: mathematical constructs that represent an arbitrary number of particles.

As a concrete example, consider an electron at rest being probed by hitting it with a photon of wavelength λ. If the photon is sufficiently energetic, its interaction with the electromagnetic field surrounding the electron may lead to the creation of an electron-positron pair. For such an interaction to occur, the photon energy must satisfy $E_\gamma = hc/\lambda > 2m_e c^2$, which implies that $\lambda < \lambda_c$. Therefore, probing the electron at scales much smaller than λ_c always risks the generation of more electrons and positrons.

A more general way to see this is directly from the uncertainty principle, rather than from any particular measurement process. Consider a particle of mass m and energy E. The uncertainly in the energy of the particle is $\Delta E = \hbar/2\Delta t \lesssim \hbar c/\Delta x$, where Δt and Δx are the typical time- and length-scales in the problem. This ΔE exceeds the pair-creation threshold, $E \sim mc^2$, provided that $\Delta x < \hbar/(mc) = \lambda$.

In summary, λ represents the smallest scale for which we can still avoid QFT. Simple quantum mechanical systems have a fixed number of particles, so we can analyze the relatively few degrees of freedom of each particle. In contrast, the excited states of a QFT can represent any number of particles. This renders QFTs especially useful for describing systems where the particle number may change over time. The price we pay is the added complexity of the theory, and infinities which are sometimes encountered when summing over the QFT degrees of freedom, and must be dealt with.

The relevance of λ for a QFT can be seen at a level much deeper than discussed above. But to show this, we first need a little more background.

E. Reminder: Special Relativity

Next, we review some aspects of special relativity (SR) which will be essential for our discussion. Wait — why? Can't we start with a non-relativistic treatment? Why are we forced to incorporate SR when discussing QFTs, and the SM in particular?

1. Motivation

Well, we already saw that even in large atoms, the inner orbitals approach relativistic velocities when the atomic number is high. Even in small atoms, a detailed spectral analysis, namely of the hyperfine structure, requires SR, because these interactions arise from relativistic effects and thus involve terms of higher order in $\beta \sim \alpha_E$ (which is why we called α_E the fine-structure constant in the first place...). As we shall see, the electroweak and QCD theories involve even higher typical velocities, so these theories are inherently relativistic. Indeed, most modern QFTs, including QED, QCD, and the entire SM, are relativistic theories.

More importantly, perhaps, the combination of SM and Quantum Mechanics (QM) alone strongly constrains the nature of interactions possible. In fact, we will later demonstrate how the Lorentz invariance of a quantum mechanical theory yields, under simple assumptions, the electric force, general relativity (GR), and more generally all the YM theories of the SM, and probably also supersymmetry (SUSY). Formally, we may say that for long-range interactions,

$$SR + QM \longrightarrow YM + GR\,(+SUSY?) \tag{5}$$

No need, therefore, for gedanken experiments involving elevators, assuming intricate inner symmetry structures, etc.!

2. The Lorentz transformation and Lorentz invariance

As a start, consider two inertial reference frames, S and S', where S' is moving with constant velocity v with respect to S. The spacetime locations of an event as measured in the S and S' frames are related by a Lorentz boost: the only possible linear transformation that preserves the speed of light in all frames. For $\vec{v} = v\hat{x}$, it may be written as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \text{ where } \Lambda = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

The inverse transformation is given by $\Lambda(\beta \to -\beta)$; verify that this yields the inverse matrix, i.e. that $\Lambda(\beta) \cdot \Lambda(-\beta) = I$, the unit matrix.

SR is constructed based on the relativity principle: the requirement that physics be the same in all inertial frames. This leads us to construct Lorentz-invariant objects, from which we can write Lorentz-invariant equations. In order to construct a Lorentz scalar, we first need to define 4-vectors:

$$A^\mu = \begin{pmatrix} A^o \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix}. \tag{7}$$

This is useless unless we define a rule for multiplying 4-vectors:

$$A \cdot A \equiv A^\mu A_\mu \equiv A_0^2 - \vec{A} \cdot \vec{A}. \tag{8}$$

Formally, we have defined the Minkowski metric $\eta_{\mu\nu}$, such that $A \cdot A = \eta_{\mu\nu} A^\mu A^\nu$, with

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

Sometimes equations (8) and (9) are defined with a minus sign; we will stick with the $(+, -, -, -)$ convention for the signature.

The Lorentz transformation for the differentials is the same as above,

$$\begin{pmatrix} c\,dt' \\ dx' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \end{pmatrix}, \tag{10}$$

which yields the velocity transformation, and so forth. We also define the proper time,

$$d\tau \equiv \frac{\sqrt{dx \cdot dx}}{c} = \sqrt{dt^2 - \frac{1}{c^2} dx^2} = \frac{dt}{\gamma}, \tag{11}$$

which it the time in the rest frame.

3. Important four-vectors

Now, as long as we construct our physical equations out of 4-scalars, 4-vector, and in general 4-tensors, Lorentz invariance is guaranteed. Let's begin with the 4-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix} = c\beta^\mu, \text{ where we defined } \beta^\mu \equiv \begin{pmatrix} \gamma \\ \gamma \vec{\beta} \end{pmatrix} . \tag{12}$$

Note that here γ and β describe the motion of a particle, and are unrelated to the boost transformation discussed earlier.

The 4-vector β^μ is very useful. First note that it has unit 4-norm, $\beta \cdot \beta = \gamma^2 - \gamma^2\beta^2 = 1$. Hence, $U^\mu U_\mu = c^2$ has a fixed 4-norm, regardless of the velocity of the object. Next, notice that in the rest frame of a particle, which we denote by primes, $\beta'^\mu = (1, \vec{0})$. Hence, for any 4-vector A^μ we always have

$$A \cdot \beta = A' \cdot \beta' = A'_0 \,, \tag{13}$$

so contracting with β^μ (in any reference frame) always extracts the rest-frame timelike component.

Using the 4-velocity, we can now define the 4-momentum:

$$p^\mu \equiv m_0 U^\mu = \begin{pmatrix} \gamma m_0 c \\ \gamma m_0 \vec{v} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = m_0 c \beta^\mu \,. \tag{14}$$

Here, m_0 is the rest mass; we do not use a velocity-dependent mass in this course. The 4-norm of the momentum is therefore $p \cdot p = (m_0 c)^2$. This is a central result: the 4-length of the momentum must be m (in $c = 1$ units). This is called the *mass shell*. For a massless particle $v = c$ and $p^\mu = (p, \vec{p})$, and $p \cdot p = 0$ designates the mass shell.

Let's remind ourselves of some additional central results. Inspection of p^μ reminds us that the spacelike component of the 4-momentum is $\vec{p} = \gamma\vec{\beta}mc = (\gamma^2 - 1)^{1/2}mc$, and that the energy is $E = \gamma m_0 c^2 = [(mc^2)^2 + (pc)^2]^{1/2}$. Recall that with the same construction as above, we can proceed and define the 4-acceleration $a^\mu \equiv dU^\mu/d\tau$, the 4-force $F^\mu \equiv m_0 a^\mu$, etc. What other important 4-vectors do you remember?

We will need the:

$$\text{4-vector potential } A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}, \text{ and 4-current } J^\mu = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} . \tag{15}$$

The inhomogeneous Maxwell equation can now be written as

$$\Box A^\mu = \frac{4\pi}{c} J^\mu \,, \tag{16}$$

where $\Box \equiv c^{-2}\partial_{t,t} - \nabla^2$ is the d'Alambert operator. Here and henceforth we use the Lorentz gauge $\partial_\mu A^\mu = 0$; recall that this does not completely fix the gauge freedom, as one can still make the gauge transformation $A^\mu \to A^\mu + \partial^\mu f$ with an arbitrary scalar function f (i.e., a function satisfying $\partial^\mu\partial_\mu f = 0$).

Q: Consider a free wave, given by $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. Clearly, the wavevector $k \sim 1/\lambda$ is not Lorentz invariant, and neither is the angular frequency $\omega = 2\pi\nu$. What is the invariant quantity in such a wave?

A: It is the phase, $\phi = \vec{k}\cdot\vec{x} - \omega t$, that is invariant, because all observers will agree about the locations of the maxima and minima of the wave. Thus, we may define a the wave 4-vector,

$$k^\mu \equiv \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} \,. \tag{17}$$

To illustrate the usefulness of this vector, consider how the frequency of the wave changes between different frames.

Q: If in our frame we measure a wave k^μ, what is the frequency measured by an observer moving at some velocity v with respect to us?

A: Using the β-contraction trick we saw earlier, denote by primes the rest frame of the observer, such that

$$\omega' = ck\cdot\beta = \omega\gamma - c\vec{k}\cdot\gamma\vec{\beta} = \gamma(\omega - \vec{k}\cdot v) = \gamma\omega\left(1 - \frac{\vec{k}\cdot\vec{v}}{\omega}\right) \,, \tag{18}$$

which is the well known Doppler effect formula. For a photon $\omega = ck$, so this becomes $\omega' = \gamma\omega(1 - \beta\cos\theta)$, where $\cos\beta \equiv \hat{k}\cdot\hat{v}$.

F. Fundamental Interactions

1. Spontaneous photon emission

We finally have all we need to discuss the fundamental forces from the perspective of particles. To get started, consider the following.

Q: Can a free particle, such as an electron, spontaneously emit a photon?
A: This is the well known process $e \to e + \gamma$, which lies at the core of the electromagnetic interactions (more about such fundamental interactions below). However, this processes appears to be forbidden by momentum-energy conservation! Consider the 4-momentum before and after the emission of the photon:

$$P^\mu = \begin{pmatrix} E/c \\ 0 \end{pmatrix} = \begin{pmatrix} m_e c \\ 0 \end{pmatrix} \; ; \; P'^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} + \begin{pmatrix} q \\ \vec{q} \end{pmatrix} = \begin{pmatrix} \sqrt{(m_e c)^2 + p^2} \\ \vec{p} \end{pmatrix} + \begin{pmatrix} q \\ \vec{q} \end{pmatrix} . \qquad (19)$$

The conservation of 3-momentum ($\mu = 1, 2, 3$) yields $0 = \vec{p} + \vec{q}$, such that $\vec{p} = -\vec{q}$ and $q = p$. But now we find from energy conservation ($\mu = 0$) that $m_e c = \sqrt{(m_e c)^2 + p^2} + p^2 > m_e c$, which is impossible for $p \neq 0$. Thus, the energy is too high after the emission, and as the photon cannot carry negative energy, we are forced to conclude that the interaction is impossible. But is it really impossible?

2. Violating the conservation laws

Clearly, such spontaneous emission is impossible, classically. But it turns out in fact to be possible quantum mechanically! The reason is that in QM, there is an inherent uncertainty in the energy and momentum. Suppose that the uncertainty in the energy, ΔE, is sufficiently large such that energy and momentum conservation can be satisfied simultaneously within this margin of error. Then a photon can indeed be emitted. But how long can this go on, or equivalently, how far can the photon travel? From the uncertainty principle, the energy offset can be sustained only over a timescale $\Delta t \sim \hbar/\Delta E$. Hence, the larger the violation of the conservation laws, the shorter the timescale over which the violation can persist, and the smaller the distance $l \sim c\Delta t \sim c\hbar/\Delta E$ over which the photon can propagate before it must undergoe some interaction (e.g., is absorbed by another electron or a nucleous) that restores the conservation of energy-momentum. In such a case we say that the photon has "burrowed energy" from the vacuum.

In the above process, if we assume that the electron obtained some 3-momentum $p > 0$, then we find a negative photon energy,

$$q_0 = m_e c - [(m_e c)^2 + p^2]^{1/2} < 0\,. \qquad (20)$$

It is useful to study the 4-momentum transfer in the interaction, defined as $q \cdot q$. In the above process, as $q_0 < 0$, it also turns out to be negative:

$$q \cdot q = q_0^2 - \vec{q} \cdot \vec{q} < 0 \,. \tag{21}$$

Now, recall that a massless particle such as a photon is expected to lie on its mass shell, $q \cdot q = 0$. Clearly, the photon spontaneously emitted by the electron does not satisfy this shell requirement. We call such a process an *"off shell"* interaction. A particle that is off-shell is called a *virtual particle*.

Now, it is precisely the exchange of such virtual particles, continuously emitted and absorbed by any charge carrier, that mediate the fundamental interactions! The description of forces through the exchange of virtual particles is a powerful theoretical tool, essential in our understanding of the SM.

3. Virtual particles and the fundamental forces

Instead of a photon, we now generalize the discussion for an arbitrary particle of mass m and 4-momentum q^μ. As Eqs. (20) and (21) are still valid, the 4-momentum transfer is negative here, too. The energy imbalance is therefore

$$\Delta E = c\sqrt{|q \cdot q - (mc)^2|} > mc^2 \,. \tag{22}$$

This means that, unlike for a massless particle, the energy imbalance cannot be arbitrarily small for a massive particle. The uncertainty principle now limits the lifetime of our virtual particle: $\Delta t \sim \hbar/\Delta E < \hbar/(mc^2)$. The distance over which our particle can propagate is thus limited by $\Delta l \le c\Delta t \le \hbar/(mc) = \lambda$. Thus we found a deeper role played by the Compton length: it is the limiting range of the virtual particle.

Quite generally, we may write the range of the virtual particle as

$$\Delta l \sim \frac{\hbar c}{mc^2} \sim \frac{10^{-27} \cdot 3 \cdot 10^{10}}{1.6 \cdot 10^{-3}} E_{GeV}^{-1} \text{ cm} \simeq 2 \cdot 10^{-14} E_{GeV}^{-1} \text{ cm} = 0.2 E_{GeV}^{-1} \text{ f} \,. \tag{23}$$

We can now reveal our plan: we wish to explore the SM and the nature of its interactions, by studying the properties of the bosons carrying each of the fundamental forces.

Let us begin with the electromagnetic interaction. It is carried by the photon, which is massless. As $m_\gamma = 0$, there is no limit to the range of the force, $\lambda_\gamma \to \infty$: indeed, this is a long range force with no characteristic lengthscale.

4. The force holding together the nucleus

Next, consider the force holding together the nuclei (protons and neutrons) in the nucleus. Which particle mediates this interaction? Clearly, it is not the photon. The culprit force is strong enough to overcome the electric repulsion between the protons, so it must involve the strong interaction. But it cannot be carried by gluons: the proton and neutron are baryons, i.e. are color neutral.

It turns out that we are dealing here with a *residual* strong force, i.e. the force generated by composite color particles: the light mesons, in particular the pion, discussed above, which is the lightest meson and has spin 0. The situation is closely analogous to the residual electromagnetic forces between electrically neutral particles such as dipoles or molecules, namely the van der Vaals forces. The mass of the pion is $m_\pi \simeq 140$ MeV. Therefore, the range of this residual strong force is $\Delta l \simeq 0.2(0.14)^{-1}$ f $\simeq 1.5$ f.

Does this explain the sizes of nuclei? You bet! The radius of the hydrogen nucleus is about 1 f, but this tells us something about the proton size rather than about the interaction between nuclei. The radius of the $^2_4$He nucleus is ~ 2 f. Roughly speaking, nuclei radii follow the approximation $r \simeq 1.2A^{1/3}$ f, where A is the mass number. This works better in heavy nuclei, for example $r \simeq 7.5$ f for $^{92}_{238}$U. The measured radii of nuceli are in good agreement with the range of the virtual pion. In fact, this is precisely what initially led to the conjecture that pions exist!

We shall continue this line of study later, although you are encouraged to stop now and figure out the ranges of the other forces in the SM. Before we proceed to do so, it is useful to examine what additional information can we infer about the properties of a force, simply by knowing the identity of the particle mediating it. For example, we know that the electric force between two charges q_1 and q_2 separated by a distance r is given by $F = q_1q_2/r^2$, such that like-particles repel each other. But could we have inferred this simply from the properties of the photon? If so, can we do the same for the other interactions?

5. Static forces from virtual particle exchange

In order to find the behavior of some force, we need to know the QM equation describing the evolution of the underlying, force-mediating boson. For a virtual spin zero particle, this is the Klein-Gordon equation, but what do we use for the photon? And for the graviton? Remember we need equations for a massive particle, because the virtual particle always has some mass. Let's briefly remind ourselves how these equations come to be.

The idea is, as usual, to take a classical equation and replace $E \to i\hbar\partial_t$ and $p \to i\hbar\nabla$. Thus, the non-relativistic relation $E = p^2/(2m)$ becomes the Schrödinger equation,

$$E = p^2/(2m) \longrightarrow i\hbar\partial_t\psi = -(\hbar^2\nabla^2/2m)\psi \longrightarrow i\partial_{x_0}\psi = -\frac{\lambda}{2}\nabla^2\psi\,. \tag{24}$$

The relativistic quadratic relation $E^2 = (mc^2)^2 + (pc)^2$ becomes the Klein-Gordon (KG) equation,

$$E^2 = (mc^2)^2 + (pc)^2 \longrightarrow -\hbar^2\frac{\partial^2\phi}{\partial t^2} = (mc^2)^2\phi - \hbar^2c^2\nabla^2\phi \longrightarrow (\Box + \lambda^{-2})\phi = 0\,. \tag{25}$$

For completeness: the relativistic relation $mc^2 = \sqrt{E^2 - (pc)^2}$ yields the Dirac equation,

$$(i\gamma^\mu\partial_\mu - \lambda^{-1})\psi = 0\,,\ \text{where the } \gamma^\mu \text{ are the four Dirac matrices [2].} \tag{26}$$

The classical Maxwell equation (16) becomes, upon adding a mass term, the Proca equation, and in the absence of source terms we obtain

$$\Box A^\mu = \frac{4\pi}{c}j^\mu \longrightarrow (\Box + \lambda^{-2})A^\mu = 0\,. \tag{27}$$

For the massive graviton, from the Einstein equations in the source free, weak field limit we find

$$(\Box + \lambda^{-2})h^{\mu\nu} = 0\,. \tag{28}$$

Note that for the bosons of interest, the relevant equations (25), (27) and (28) are simply KG equations that act separately on each component of the boson field. In the massless limit, or equivalently on scales $\ll \lambda$ in which the λ^{-2} term is negligible, we obtain free boson solutions

$$\phi = e^{ik\cdot x}\,, A^\mu = \epsilon^\mu e^{ik\cdot x}\,,\ \text{and } h^{\mu\nu} = \epsilon^{\mu\nu}e^{ik\cdot x}\,. \tag{29}$$

Here, ϵ^μ and $\epsilon^{\mu\nu}$ are polarization tensors, which will play an important role when we study the structure of the SM.

[2] To see this, Dirac postulated that $\sqrt{\frac{1}{c^2}\partial_t^2 - \nabla^2} = A\partial_x + B\partial_y + C\partial_z + D\partial_t$. There is no solution for A, B, C, D if they are numbers. However, if they are allowed to be matrices, then one does obtain a solution: $A = -i\gamma^{(0)}\gamma^{(1)}, B = -i\gamma^{(0)}\gamma^{(2)}, C = -i\gamma^{(0)}\gamma^{(3)}, D = -i\gamma^{(0)}$.

Now we return to the task at hand: find the form of the force mediated by a given virtual boson. We are looking for the *static* force between stationary charges (those charges that couple to this gauge boson: electric charge for the photon, mass for the graviton, etc.). Therefore, we neglect time-derivatives. For simplicity, consider a spin 0 field, i.e. the KG equation. Here

$$(\Box + \lambda^{-2})\phi = 0 \longrightarrow (-\nabla^2 + \lambda^{-2})\phi = 0 \longrightarrow \frac{1}{r^2}\partial_r(r^2\partial_r\phi) = \frac{\phi}{\lambda^2} \longrightarrow \phi \propto \frac{1}{r}e^{-r/\lambda}. \tag{30}$$

This is the Yukawa potential, thought to be responsible for example for holding the nucleus together, through the exchange of virtual pions.

G. Mathematical structure and Feynman diagrams

Next, we outline the theoretical structure underlying QFT and the SM in particular. An appropriate discussion of this topic would require a full-scale course with considerable mathematical excursions and substantial practice. So, we resort to simply sketching the basic ideas, with no proof, behind YM theories, the path integral formalism, and Feynman diagrams. Towards the end we highlight what we need to know, practically, in order to be able to compute order-of-magnitude estimates.

1. The gauge symmetry fixes the interactions

We start with the Lagrangian (actually, the Lagrangian density) of some particle, for example the Dirac Lagrangian of a spin 1/2 fermion, $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - \lambda^{-1})\psi$. Now, $\mathcal{L}$ is invariant under a global $U(1)$ symmetry, $\psi \to e^{i\phi}\psi, \bar{\psi} \to e^{-i\phi}\bar{\psi}$, where ϕ is a constant. Next, we conjecture that in fact there exists a local symmetry, i.e. that $\phi = \phi(x)$ can be a local phase. The λ^{-1} term (in fact a mass term) is invariant under this transformation, but the derivative term is not. This forces us to introduce some tool to convey the information about the different phases throughout spacetime, and this is the role of the gauge boson. Mathematically, we replace ∂_μ with a covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, and we're fine. However, this means we introduced a new field A, so we must endow it with a kinetic term. It turns out that we have no choice but to add a Lagrangian term of the form $-F^{\mu\nu}F_{\mu\nu}/4$, where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. Surprise — this term precisely gives the Maxwell equations, although we were not aiming for them!

What about other, more complicated gauge symmetries? Here multiple phases must be defined, the covariant derivative is more complicated, and so is the kinetic term we must add for the fields introduced in the covariant derivative. But the logic remains unchanged. Each gauge group can be described in terms of its own electric and magnetic fields. However, for non-abelian groups, these equations become non-linear and difficult to solve.

2. How this lets us compute probabilities

Back to the simple symmetry with a single phase. Our modified Lagrangian now has a term of the form $e\bar{\psi}\gamma^\mu A_\mu \psi$, due to the covariant derivative. Now we would like to use the Lagrangian to compute the probabilities of various processes by some computational tool. One method to do so is the path integral formalism, which may be thought of as a generalized Hamilton action principle: it involves integrating $e^{i\int_a^b L dt}$, but not just over the paths from a to b; rather, the integral is carried out over all possible field configurations in spacetime. Without going into any detail concerning this beautiful but lengthy formalism, it is clear that expanding the exponent will produce the above $e\bar{\psi}A\psi$ term, as a contribution to the amplitude $\mathcal{M}$ that enters the final probability as $|\mathcal{M}|^2$. But it will also produce high order terms, $e\bar{\psi}A\psi + (e\bar{\psi}A\psi)^2 + \ldots$. Each such term corresponds to an interaction involving 3, 6,... particles, and each is associated with a Feynman diagram. Some examples are shown in the figures below, and will be discussed next time.

In general, Feynman diagrams are a pictorial representations of the mathematical expressions governing the behavior of subatomic particles. The Feynman diagrams provide a tool for carefully computing all the contributions to the amplitudes of a certain interaction, but these calculations are too laborious for this course. We will simply use the diagrams to keep track of the various interactions recognized by the SM, and only estimate to an order of magnitude the probabilities of these processes, typically the cross sections for collisions and the decay rates of particles.

3. What we need to know at this point

Each term of the form $e\bar{\psi}A\psi$ can be represented as a three-way encounter between particles or anti-particles. We draw it as two lines representing the fermion, and a wiggly line representing the boson, all three meeting at a point called a vertex — see Figure 2. To distinguish between ψ and $\bar{\psi}$, we draw arrows along the solid lines, say an ingoing arrow for the ψ and an outgoing arrow for the $\bar{\psi}$. Now, our 2D diagram lies in the phase space of space vs. time, but we are free to draw the lines and the axes at any timelike or lightlike angles we want. This means that with the same diagram, an ingoing arrow can represent an incoming fermion, or equivalently an outgoing anti-fermion! The wiggly line stands for either an incoming or an outgoing boson, etc. Thus, our three-point diagram can represent a fermion spontaneously emitting a boson, or a fermion and an anti-fermion annihilating and thus producing a boson, or a boson decaying into a fermion-anti-fermion pair!

A few comments are in place. First, you may have already guessed that at least one of the arrows in our diagram must stand for a virtual particle (e.g., spontaneous emission is not classically possible without some additional interaction), so we are not done yet. Indeed, Figure 2 represents only one half of the full diagram. We must combine it with another similar diagram, for example one in which the boson wiggly line terminates, leaving two additional fermion lines, as in the diagrams shown in Figure 3. Here the virual particle is the boson, created and annihilated as the particles apply a force on each other, or alternatively are annihilated and created in pairs - depending on how you draw the lines! Can a fermion be the virtual particle? Equivalently, can you combine two diagrams such that the fermion is created and annihilated? Sure, simply draw a diagram with two vertices connected by a fermion line, representing a boson absorbed by the fermion (which thus becomes temporarily off-shell!) and later emitted. As a rule, any closed particle line, created at some vertex and vanishing at another, represents a virtual particle.

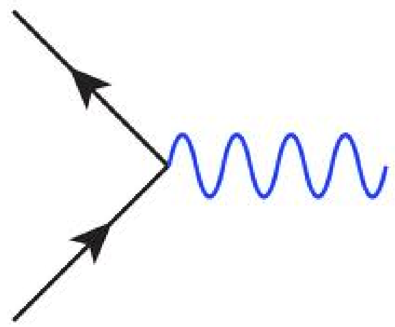

FIG. 2: A Feynman diagram of order $\bar{\psi}A\psi$. (Figure from Wikipedia.)

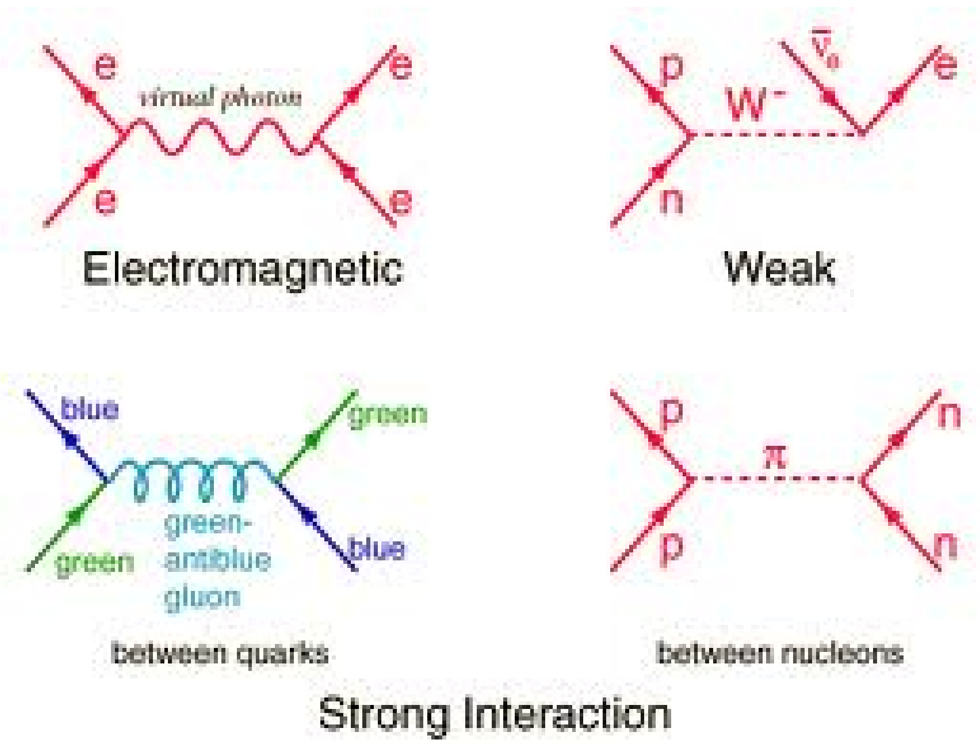

FIG. 3: Feynman diagrams of order $(\bar{\psi}A\psi)^2$ for the different fundamental interactions. (Figure from Wikipedia.)

Next, consider the conservation laws at the vertex, for example the ingoing vs. outgoing charge. If we draw the diagram in Figure 2 such that it represents a fermion and anti-fermion annihilating, then the ingoing charge is zero, and so it the outgoing charge. This is clearly true for any other interpretation of the diagram, so we can say that the vertex is charge neutral. Is this true for all conserved charges, say electric charge, strong charge, lepton number (+1 for leptons, -1 for anti-leptons), baryon number, etc.? Clearly, yes: the vertex is neutral for all symmetries of the Lagrangian, because it represents a term in $\mathcal{L}$ which must be a singlet state. What about continuous symmetries, for example ingoing vs. outgoing momentum-energy? The same: momentum and energy are always conserved, which forces (at least) one of the particles to go off-shell. What about the spin state of the vertex? The same: it must be a singlet state, so we cannot have a vertex connecting, say, three fermions.

Now, for the million dollar question: what is the value of $\mathcal{M}$, in other words what is the probability associated with the process? Unfortunately, the answer can be excruciatingly difficult to obtain, involving multiple, sometimes apparently diverging integrals. But in many cases we will nevertheless be able to obtain an order of magnitude estimate, which is all we really need in this course. As each vertex represents a term $e\bar{\psi}A\psi$, it contributes a factor $\propto e^2 \propto \alpha$ to the final, $|\mathcal{M}|^2$ probability. The coupling constant involved depends on the interaction: α_E if a photon is involved, α_W if a W or Z boson is involved, etc. Hence, the probability of two particles scattering each other through the emission and absorption of a virtual particle will always be proportional to the relevant α^2, because two vertices are involved. A decay of a particle intro three particles, such as β decay, will be proportional to the relevant α^2. And so forth. In many cases, we can roughly estimate the probability from dimensional analysis, so knowing the α dependence helps us fix the numerical coefficient!

One last comment, which will become important later. The above discussion makes our analysis explicitly *local*. Why? We assigned the interaction vertex to a specific point in spacetime, so we implicitly assumed that the interaction between particles takes place at a given point! We shall find out later that this leads to a whole bag of worms. In order to fix this, we will have to venture out to the relatively uncharted waters beyond the SM, to the realm of string theory, holographic theories, etc.