2 Applying extreme cases

2.1 Rutherford scattering

Let an electron pass an ion with the velocity v at the impact parameter b. If the deviation angle is small, the equation of motion for the transverse velocity is reduced to

$$m\dot{v}_\perp = F_\perp = \frac{Ze^2}{r^2}\cos\alpha = \frac{Ze^2 b}{(b^2+v^2t^2)^{3/2}}. \tag{2.1}$$

This yields the deviation angle

$$\theta = v_\perp/v = \frac{Ze^2}{mv}\int_{-\infty}^{\infty}\frac{bdt}{(b^2+v^2t^2)^{3/2}} = \frac{2Ze^2}{mbv^2}. \tag{2.2}$$

Note that to within a factor of 2, one can find a rough estimate as $v_\perp \sim F_\perp \Delta t/m$, where $F_\perp \sim Ze^2/b$ and $\Delta t \sim v/b$. The angle is small provided $Ze^2/b \ll m_e v^2$, i.e. if the potential energy is small as compared with the kinetic energy. The electron trajectory is strongly distorted if $b \sim Ze^2/m_e v^2$.

One can conveniently characterize the collision process by the differential collision cross section, which is defined such that if the particle flux F passes the scattering center, the number of particles scattered to within the solid angle $d\Omega$ per unit time is $\dot{N} = F d\sigma$. In our case, the electrons are scattered to within the solid angle $d\Omega$ around the angle θ pass in the ring $2\pi b db$, so that $\dot{N} = F 2\pi b db$ and $d\sigma = 2\pi b db$. Making use of eq. (2.2) one finds

$$d\sigma = 2\pi b\frac{db}{d\theta}d\theta = 2\pi\left(\frac{2Ze^2}{mv^2}\right)^2\frac{d\theta}{\theta^3} = \left(\frac{2Ze^2}{mv^2}\right)^2\frac{d\Omega}{\theta^4}, \tag{2.3}$$

which coincides, at a small θ, with the Rutherford cross section.

2.2 Radiation

Emission of waves is described by the wave equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 4\pi\rho, \tag{2.4}$$

where φ is an appropriate wave variable, c the wave speed, $\rho(\mathbf{r},t)$ the source. For example, sound is the density (pressure) wave therefore it is described by the scalar equation. Electro-magnetic waves are described by 4-vector potential whereas gravity waves are described by the metric tensor.

For monochromatic sources, the wave equation is reduced to the Helmholtz equation

$$\Delta\varphi + \frac{\omega^2}{c^2}\varphi = -4\pi\rho. \tag{2.5}$$

In the near zone, $r \ll c/\omega$, one can neglect the second term in the lhs. Then the field is described by the solution to the Poisson equation

$$\varphi = \int\frac{\rho(\mathbf{r}',t)d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \ldots \tag{2.6}$$

The multipole expansion is justified provided the size of the source, a, is much less than the wavelength, $a \ll c/\omega$, which implies that motions within the source are slow as compared with the wave speed, $v/c \sim \omega a/c \ll 1$.

At large distances, $r \gg c/\omega$, the equation describes radially outgoing waves (the wave zone):

$$\varphi = \frac{G}{r} e^{-i\omega(t-r/c)}. \tag{2.7}$$

The wave amplitude, G, may be found by matching (2.7) and (2.6) in the intermediate zone, $r \sim c/\omega$.

In **sound waves**, the gas moves under the pressure gradients, $\mathbf{v} \propto \nabla p$ (the exact relation is $i\omega\rho\mathbf{v} = \nabla p$), therefore one can define the velocity potential as $\mathbf{v} = \nabla\varphi$ and consider φ as the wave variable in eq. (2.4). Then the first term in (2.6) describes radial motion with the velocity

$$v = \frac{\partial\varphi}{\partial r} = -q/r^2. \tag{2.8}$$

The velocity distribution (2.8) implies $\nabla \cdot \mathbf{v} = 0$ so that the flow in the near zone is incompressible. The mass flux via the sphere of the radius r is $F = 4\pi r^2 \rho v = -4\pi\rho q$ therefore the volume of the gas within the sphere varies with the rate $-F/\rho = -4\pi r^2 v = 4\pi q$, which implies that the source volume varies (e.g., a speaker or a drum) with the rate $\dot{V} = -4\pi q$. Matching with eq. (2.7) at $r \sim c/\omega$ yields

$$Ge^{-i\omega t} = q = -\dot{V}/4\pi = \frac{i\omega\delta V}{4\pi} e^{-i\omega t}, \tag{2.9}$$

where δV is the amplitude of the source volume variation, such that $\dot{V} = i\omega\delta V$. Now the velocity in the far zone is found as

$$v = \frac{\partial\varphi}{\partial r} = -\frac{\omega^2\delta V}{4\pi c r} e^{-i\omega(t-r/c)}. \tag{2.10}$$

The energy flux in the the sound wave, S, is the wave speed by the energy density, the last being twice the kinetic energy density. Therefore the emitted power is finally found as

$$Q = 4\pi r^2 \overline{S} = 4\pi r^2 \rho\overline{v^2} c = \frac{1}{4\pi c}\rho\overline{\ddot{V}^2}. \tag{2.11}$$

If $\dot{V} = 0$ (sound is emitted by a body oscillating as a whole), one has to match eq. (2.7) with the second term in eq. (2.6).

The **electromagnetic waves** are described by four wave equations for components of 4-potential, or, which is the same, by six equations for the components of the electric and magnetic fields. Let us take into account that in the wave zone, $E = B$ and

$$\mathbf{E} = \frac{\mathbf{G}}{r} e^{-i\omega(t-r/c)}, \tag{2.12}$$

where $\mathbf{G} \cdot \mathbf{r} = 0$. Then for rough estimate, one can match only the electric field. In the near zone, eq. (2.6) yields

$$E = \frac{q}{r^3}\mathbf{r} + \frac{3(\mathbf{r}\cdot\mathbf{d})\mathbf{r} - r^2\mathbf{d}}{r^5} + \dots. \tag{2.13}$$

The first term is longitudinal and moreover, it does not depend on time; therefore it could not be matched with the wave solution. The second term could be matched with the wave solution provided $d = d_0 \exp(-i\omega t)$. Then one finds roughly

$$G \sim \frac{\omega^2}{c^2} d_0. \tag{2.14}$$

The radiation power is estimated as $P \sim 4\pi r^2 S$, where $S = c\overline{E^2}/4\pi$ is the Poynting flux. Then one finally gets

$$P \sim 4\pi r^2 S \sim cG^2 \sim \frac{\ddot{d}^2}{c^3}, \tag{2.15}$$

in accord with Larmor's formula. If $d = 0$, one has to expand (2.6) to the quadruple term and also take into account the magnetic dipole contribution.

2.3 Forced oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{f}{m}\cos\omega t. \tag{2.16}$$

Generally one can expect that the response to the periodic force is also periodic with the same frequency, $x = A\cos(\omega t + \phi)$. We consider the most interesting case when the friction is small so that $\gamma \ll \omega_0$, ω. There are still a few extreme cases.

1. In the case $\omega \ll \omega_0$, the derivatives are small therefore one comes to a static case:

$$\omega_0^2 x = \frac{f}{m}\cos\omega t \tag{2.17}$$

so that $A = f/m\omega_0^2$, $\phi = 0$; the particle moves in phase with the force. More generally the response to any slowly varying force is $x(t) = F(t)/m\omega_0^2$.

Even more generally, in order to find a response of a system on a slowly varying external perturbation (slowly as compared with proper frequencies of the system), one has to just find the response on a static perturbation. For example, scattering of low frequency radiation on atoms is determined by their static polarizability. Namely, if the induced electric dipole momentum of an atom in a static electric field is $p = \alpha E$, then a low frequency electromagnetic wave, $E \propto \exp(i\omega t)$, induces the momentum $p \propto \exp(i\omega t)$ therefore the atom emits radiation with the power

$$P = \frac{2}{3}\frac{\overline{\ddot{p}^2}}{c^3} = \frac{1}{3}\frac{\alpha^2\omega^4 E^2}{c^3}. \tag{2.18}$$

This means that the radiation is scattered by the atom (**Rayleigh scattering**). Taking into account that the energy flux in the incident wave is $S = (1/4\pi)c\overline{[E(t)]^2} = cE^2/8\pi$, one can conveniently introduce the scattering cross section,

$$\sigma = \frac{P}{S} = \frac{8\pi}{3}\frac{\alpha^2\omega^4}{c^4}. \tag{2.19}$$

2. In the case $\omega \gg \omega_0$, one can consider oscillations as free:

$$\ddot{x} = \frac{f}{m} \cos \omega t; \tag{2.20}$$

$$x = -\frac{f}{m\omega^2} \cos \omega t \tag{2.21}$$

so that $A = f/m\omega^2$, $\phi = \pm\pi$; the particle oscillates in the opposite phase with the force.

These two extreme cases give roughly the same amplitude at $\omega \sim \omega_0$ therefore one could expect a smooth transition from $\omega \ll \omega_0$ to $\omega \gg \omega_0$. However, this is the case only if the friction force is not small. At small friction, $\gamma \ll \omega_0$, a resonance occurs. The energy considerations could help to see that the transition is not smooth.

The average power of the external force is

$$\overline{P_{\rm ext}} = \overline{\dot{x}F} = -\omega A f \overline{\sin(\omega t + \phi) \cos \omega t} = -\frac{1}{2} \omega A f \sin \phi. \tag{2.22}$$

In a frictionless system, the average power should vanish, which means either $\phi = 0$ or $\phi = -\pi$. Therefore at $\gamma = 0$, there is no smooth transition from $\omega \ll \omega_0$, when $\phi = 0$, to $\omega \gg \omega_0$, when $\phi = -\pi$. At $\gamma \neq 0$, the external power (2.22) should be equal to the dissipated power,

$$\overline{P_{\rm diss}} = m\gamma \overline{(\dot{x})^2} = \frac{1}{2} m\omega^2 \gamma A^2. \tag{2.23}$$

Equating (2.22) and (2.23), one sees that when ϕ veries from 0 to π at the transiiton from $\omega \ll \omega_0$ to $\omega \gg \omega_0$, a very large oscillaiton amplitude is achieved; the maximal amplitude is

$$A = \frac{f}{m\gamma\omega}. \tag{2.24}$$

The maximal amplitude diverges at $\gamma \to 0$. At a small γ, the friction term in Eq. (2.16) is non-negligible only if the first and the third terms in the lhs cancel each other, i.e. at $\omega = \omega_0$. Then the solution is $x = (f/m\gamma\omega_0) \sin \omega_0 t = (f/m\gamma\omega) \cos(\omega_0 t - \pi/2)$, in accord with the result of the energy consideration. The width of the resonance is determined from the condition that the difference between the first and the third terms, $(\omega^2 - \omega_0^2)x \approx 2\omega\delta\omega x$, becomes comparable with the second term, $\gamma\omega x$, which implies

$$\delta\omega \sim \gamma. \tag{2.25}$$

Balancing the external and the dissipation powers, one could find the resonance amplitude even if the dissipation could not be described by a simple friction force, $f_{\rm fr} = -\kappa v$. Let us consider scattering of electro-magnetic waves by a charged oscillator. In this case, the dissipation is due to emission of the oscillator itself; one can say that the oscillator scatters the incident radiation. Having in mind a bound electron, we take $m = m_e$, $f = eE$. Making use of Larmor's formula, one finds the average emitted power as

$$P_{\rm rad} = \frac{1}{3} \frac{e^2 \omega^4 A^2}{c^3}. \tag{2.26}$$

At the resonance, the oscillation amplitude is found from the balance of the powers (2.22) and (2.26):

$$A = \frac{3}{2} \left(\frac{c}{\omega}\right)^3 \frac{E}{e}. \tag{2.27}$$

Substituting this amplitude back to eq. (2.26), one finds the power of the scattered radiation. The scattering cross section is presented as

$$\sigma = P_{\rm rad}/S = \frac{8\pi}{3}\frac{e^2A^2\omega^4}{c^4E^2} = 6\pi(c/\omega)^2 = (3/2\pi)\lambda^2. \tag{2.28}$$

Note that this result is independent of e and m; it is of general nature and is independent of the interaction mechanism. Taking into account that the size of the emitting system is of the order of $\sim v/\omega \sim \lambda v/c$, one sees that the resonance cross section exceeds the geometrical cross section of the system roughly $(c/v)^2$ times.

In the above considerations, we addressed steady oscillations. Far from the resonance, the time necessary to reach the steady oscillations is of the order of $1/\omega$. At the resonance, the growth of amplitude may be found from the energy balance

$$\frac{dE}{dt} = \overline{P_{\rm ext}}. \tag{2.29}$$

Here $E = m\overline{\dot{x}^2} = (1/2)m\omega^2A^2$ is the oscillator energy. Substituting (2.22), one sees that at an appropriate phase shift, $0 < \varphi < \pi$, the amplitude grows linearly with time and reaches (2.24) at $t = 1/\gamma$, which is equal to the oscillator decay time.

2.4 Optical properties of materials

When an electromagnetic wave propagates through a medium, the electrons experience forced oscillations thus producing electric currents, $\mathbf{j} = eN\mathbf{v}$. This current affects the properties of the wave. Assuming that a monochromatic wave, $\sim \exp(-i\omega t + ikx)$, propagates in the x direction, one reduces the Maxwell equations,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{2.30}$$

to

$$kE = \frac{\omega}{c}B; \tag{2.31}$$

$$-ikB = \frac{4\pi}{c}j - i\frac{\omega}{c}E, \tag{2.32}$$

where $\mathbf{j} \| \mathbf{E} \perp \mathbf{B}$; $\mathbf{k} \perp \mathbf{E}, \mathbf{B}$. In vacuum, $j = 0$, which yields $\omega = ck$, $E = B$.

In **dielectrics**, all electrons are bound. If the wave frequency is smaller than characteristic frequencies in the atoms/molecules (the lasts are typically in the UV band), the electron clouds oscillate together with the field of the wave (recall eq. (2.17)) thus producing variable electric polarization $\mathbf{p} = \alpha\mathbf{E}$, where α is the atomic polarizability, which is of the order of the atomic volume, $\alpha \sim a^3$ (see question 1 in the Problem set 7).

When the polarization varies, the electric current arises. Presenting the electric dipole moment of the atom as a product of an effective charge by an effective shift, $p = qx$, one can write the electric current density as

$$j = Nq\dot{x} = N\dot{p} = -i\omega N\alpha E, \tag{2.33}$$

where N is the density of the atoms. Now eq. (2.32) may be written in the form

$$kB = \frac{\varepsilon\omega}{c} E, \tag{2.34}$$

where $\varepsilon = 1 + 4\pi N\alpha$ is the dielectric constant. Now one finds $\omega = ck/n$, where $n = \sqrt{\varepsilon}$ is the refraction index. One concludes that in gases, $n - 1 = 2\pi\alpha N \sim 2\pi a^3 N \ll 1$ whereas in condensed dielectrics, $n - 1 \sim O(1)$.

In **conductors** (plasma, metals) free electrons and the field oscillate in antiphase (see eq. (2.21)) therefore

$$j = -eNv = i\frac{e^2 N}{m_e \omega} E. \tag{2.35}$$

Then eq. (2.32) is written in the form of eq. (2.34) with the dielectric constant

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}, \tag{2.36}$$

where $\omega_p = \sqrt{4\pi e^2 N/m_e}$ is the plasma frequency. One sees that in this case, the wave phase velocity, $\omega/k = c/\sqrt{\varepsilon}$, exceeds the speed of light (check that the group velocity remains subluminal) and at $\omega < \omega_p$, the wave could not propagate. This explains high reflectivity of metals in the optical band as well as reflection of long wavelength radio waves from the ionosphere.