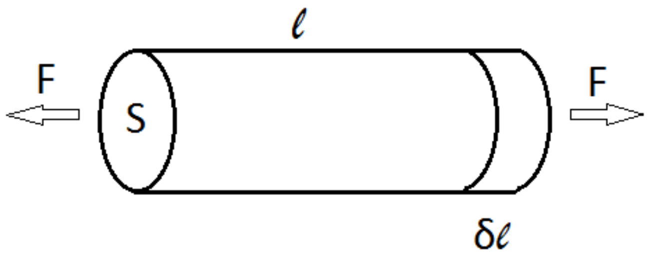
Elasticity: Stretching and Shearing

Suppose a rock is thrown at a wall. How can we find the force it will experience on impact? What would the characteristic duration of the collision be?

$$\vec{F} = \frac{\Delta(mv)}{\Delta t}$$

We can begin to estimate this by approximating a cylinder with force applied at the ends:



$F = -k\delta l$, where δl is the change in length experienced during the collision.

$k = E \cdot \frac{S}{l}$, where E is the Young Modulus of the material. $[E] = \frac{N}{m^2} = \frac{dyne}{cm^2}$

$F = -\frac{ES}{l} \cdot \delta l$

$\frac{F}{S} = -E\frac{\delta l}{l}$

1. *Young Modulus for various materials:*

Aluminum/Glass: $E = 70 \cdot 10^9 Pa$

Copper: $E = 100 \cdot 10^9 Pa$

Steel: $E = 200 \cdot 10^9 Pa$

Diamond: $E = 1200 \cdot 10^9 Pa$

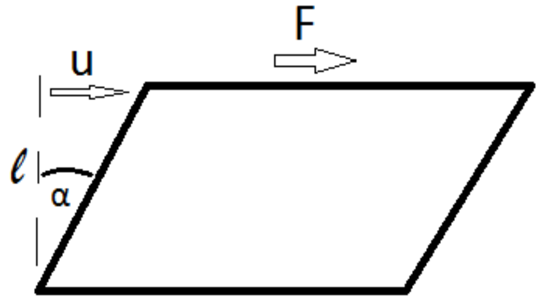
Rubber: $E = 0.1 - 0.01 \cdot 10^9 Pa$

Looking at the energy:

$$U = k\frac{\delta l^2}{2} = \frac{1}{2}ESl\left(\frac{\delta l}{l}\right)^2 \quad \Rightarrow \quad \frac{U}{V} = \frac{1}{2}E\left(\frac{\delta l}{l}\right)^2$$

Energy density is often more convenient in calculations.

A different way to apply the force:



Here,

$$\varepsilon_{xy} = \frac{u}{l} \qquad \frac{F}{S} = -G\frac{u}{l} \qquad \frac{U_{shear}}{V} = \frac{1}{2}G\left(\frac{u}{l}\right)^2$$

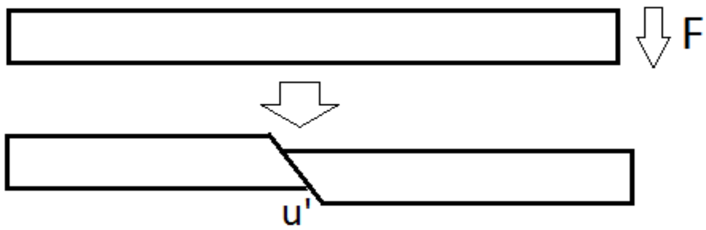
Where G is the material's Shear Modulus. Several examples:

Glass: $G = 25 \cdot 10^9 Pa$

Copper: $G = 40 \cdot 10^9 Pa$

Steel: $G = 80 \cdot 10^9 Pa$

Applying the force in a different direction will create a Shear in the material.



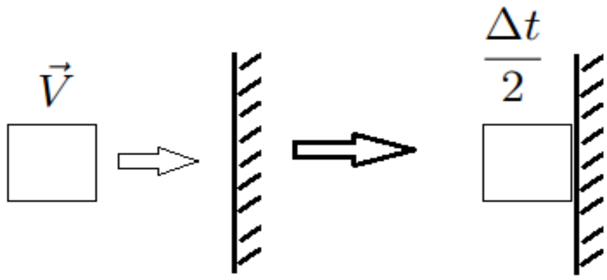
Here, $u' = \sqrt{2}u$.

$$\frac{1}{2}G\left(\frac{u'}{l}\right)^2 = \frac{1}{2}E\left(\frac{u}{l}\right)^2 \quad \Rightarrow \quad E \approx 2G(1+V)$$

Elastic theory: Applying force on a material will make all particles move together. Elasticity is when some particles move more than others.

$$\varepsilon = \frac{\partial \vec{U}}{\partial \vec{r}} \qquad \vec{\sigma} = \frac{\vec{F}}{\vec{S}}$$

Going back to the problem of the wall collision, we can divide the event into two parts: Impact and Rebound.



What's the deformation that happens to the object in the collision?

$$\delta l = \frac{v}{2} \cdot \frac{\Delta t}{2}$$

The speed v here is an inaccurate, averaged estimation.

$$F = \frac{ES}{l}\delta l = \frac{2mv}{\Delta t}$$

$$\frac{ES}{l} \cdot \frac{v\Delta t}{4} = \frac{2mv}{\Delta t}$$

We can substitute $m = S \cdot l \cdot \rho$ and get:

$$\frac{ES}{l} \cdot \Delta t = \frac{8}{\Delta t}\rho Sl \quad \Rightarrow \quad \Delta t \approx l \cdot 3\sqrt{\frac{\rho}{E}} = 3l\sqrt{\frac{\rho}{E}}$$

For example, a rock with $l = 2 \cdot 10^{-2}m, \quad \rho = 3 \cdot 10^3 \frac{kg}{m^3}, \quad E = 40 \cdot 10^9 Pa$ we get:

$$\Delta t = 3 \cdot 2 \cdot 10^{-2}\sqrt{\frac{3 \cdot 10^3}{40 \cdot 10^9}} \approx 3 \cdot 2 \cdot 10^{-2} \cdot 3 \cdot 10^{-4} \approx 10^{-5} = 10\mu s$$

F. Hearing

The range of Human hearing extends between $20Hz - 20,000Hz$.

The middle is at about $\sqrt{20 \cdot 20,000} = 2\sqrt{10^5} \sim 600Hz$

Tuning forks are musician tools that have a specific frequency in which they vibrate to create a single tone. $A \rightarrow 440Hz, \quad C \rightarrow 256Hz$.

The rods on the fork are in essence oscillators that resonate in a specific frequency. Let l be the length of the rods at the top of the fork and S the surface of the top of the rods.

$$U = \frac{1}{2}G\alpha^2 \cdot S \cdot l$$

$$\tau = -\frac{\partial U}{\partial \alpha} = -GSl\alpha$$

Where τ is the return moment of the rods.

$$\tau = I\ddot{\alpha}, \qquad I = \frac{Ml^2}{3}, \qquad M = \rho Sl$$

$$\ddot{\alpha} + \frac{GSl}{I}\alpha = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{GSl \cdot 3}{\rho S l^3}} = \frac{1}{l}\sqrt{\frac{3G}{\rho}}$$

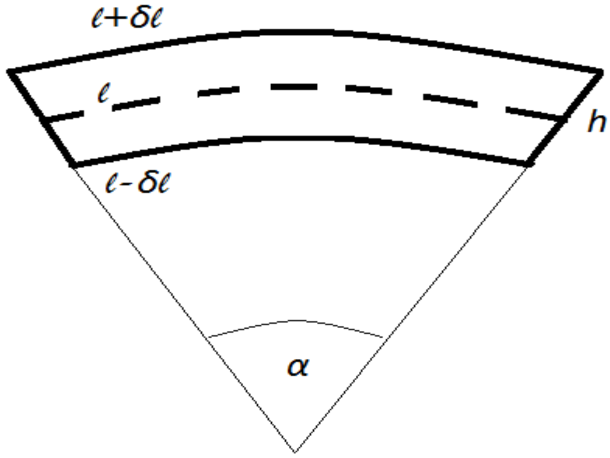
$$\nu = \frac{1}{2\pi}\omega = \frac{1}{2\pi l}\sqrt{\frac{3G}{\rho}}$$

In the class example, $l = 16cm, \quad G_{aluminum} = 25 \cdot 10^9 Pa$:

$$\nu = \frac{1}{2\pi \cdot 16 \cdot 10^{-2}} \cdot \sqrt{\frac{3 \cdot 25 \cdot 10^9}{3 \cdot 10^3}} = \frac{1}{2\pi \cdot 16 \cdot 10{-2}} \cdot 5 \cdot 10^3 \approx 10^4 Hz$$

Our frequency didn't end up with any dependence on the surface. Suspicious.

Let's take a look at the rod's deformation:



$$l = \alpha R, \quad l + \delta l = \alpha(R + \frac{h}{2}) \quad \Rightarrow \quad \delta l = \frac{\alpha h}{2}$$

This deformation is a change of the bar's length - Young Modulus.

$$U = \frac{1}{2}E\left(\frac{\delta l}{l}\right)^2\left(\frac{1}{2}\right)V = \frac{1}{16}E\left(\frac{h}{l}\right)^2 V\alpha^2$$

$$E_k = \frac{1}{2}I\left(\dot{\alpha}\right)^2$$

$$I = \frac{Ml^2}{3} = \frac{\rho V l^2}{3}, \quad \omega^2 = \frac{1}{8}E\left(\frac{h}{l}\right)^2\frac{V}{I} = \frac{1}{8}E\left(\frac{h}{l}\right)^2 \cdot \frac{3V}{\rho V l^2}$$

$$\nu = \frac{\omega}{2\pi} = \frac{1}{4\pi\sqrt{2}}\frac{h}{l^2}\sqrt{\frac{3E}{\rho}} = \frac{1.2}{4\pi} \cdot \frac{10^{-2}}{256 \cdot 10^{-4}} \cdot \sqrt{\frac{70 \cdot 10^9}{3 \cdot 10^3}} \simeq 10^-1 \cdot \frac{10^{-2} \cdot 4}{10^3 \cdot 10^{-4}} \approx 200Hz$$