Gravity 2

1 Set 1, question 6

The line element is given as

$$ds^2 = -dt^2 + dx^2 + 2a^2\left(t\right)dxdy + dy^2 + dz^2.$$

1.1 $g_{\mu\nu}, g^{\mu\nu}, D_\mu, R^\mu{}_{\nu\rho\sigma}, R_{\mu\nu}, R$

1.1.1 The metric $g_{\mu\nu}$

From the scalar line element one can obtain a matrix bilinear form:

$$ds^2 = \begin{pmatrix} dt & dx & dy & dz \end{pmatrix} \overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & a^2(t) & 0 \\ 0 & a^2(t) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{g_{\mu\nu}} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}.$$

1.1.2 The inverse metric $g^{\mu\nu}$

Inversion of a block-diagonal matrix can be carried by inverting each individual block. The middle 2×2 block is inverted according to the rule

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

One obtains the inverse metric

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-a^4(t)} & \frac{-a^2(t)}{1-a^4(t)} & 0 \\ 0 & \frac{-a^2(t)}{1-a^4(t)} & \frac{1}{1-a^4(t)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.1.3 Covariant derivatives

The covariant derivative of some vector V^ν is

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda,$$

where the Christoffel symbol is defined as

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\nu\alpha} \left(g_{\mu\alpha,\lambda} + g_{\lambda\alpha,\mu} - g_{\mu\lambda,\alpha} \right).$$

The metric is independent of x, y, z, so the only non vanishing derivatives of the metric are the ones w.r.t t. The only non-vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^t_{xy} &= \frac{1}{2} g^{tt} \left(-g_{xy,t} \right) = a\partial_t a, \\
\Gamma^x_{xt} &= \frac{1}{2} g^{xy} g_{xy,t} = \frac{a^3 \partial_t a}{a^4 - 1}, \\
\Gamma^y_{xt} &= \frac{1}{2} g^{yy} g_{xy,t} = \frac{a \partial_t a}{1 - a^4}, \\
\Gamma^x_{yt} &= \frac{1}{2} g^{xx} g_{yx,t} = \frac{a \partial_t a}{1 - a^4}, \\
\Gamma^y_{yt} &= \frac{1}{2} g^{yx} g_{yx,t} = \frac{a^3 \partial_t a}{a^4 - 1}.
\end{aligned}$$

The Covariant derivatives are then

$$\begin{aligned}
D_t V^\nu =& \partial_t V^\nu + \left[\frac{-a^3 \partial_t a}{1 - a^4} \delta^\nu_x + \frac{a \partial_t a}{1 - a^4} \delta^\nu_y \right] V^x \\
& + \left[\frac{a \partial_t a}{1 - a^4} \delta^\nu_x + \frac{a^3 \partial_t a}{a^4 - 1} \delta^\nu_y \right] V^y, \\
D_x V^\nu =& \partial_x V^\nu + \left[\frac{-a^3 \partial_t a}{1 - a^4} \delta^\nu_x + \frac{a \partial_t a}{1 - a^4} \delta^\nu_y \right] V^t \\
& + a \partial_t a \delta^\nu_t V^y, \\
D_y V^\nu =& \partial_y V^\nu + \left[\frac{a \partial_t a}{1 - a^4} \delta^\nu_x + \frac{a^3 \partial_t a}{a^4 - 1} \delta^\nu_y \right] V^t \\
& + a \partial_t a \delta^\nu_t V^x, \\
D_z V^\nu =& \partial_z V^\nu.
\end{aligned}$$

1.1.4 Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$

The Riemann tensor is defined through Christoffel symbols as

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho}.$$

Based on the non-trivial Christoffel symbols, the only non-vanishing components of the Riemann tensor are

$$\begin{aligned} R^{t}{}_{xyt} = -R^{t}{}_{xty} = R^{t}{}_{yxt} = -R^{t}{}_{ytx} &= -\partial_t \Gamma^{t}_{xy} + \Gamma^{t}_{xy}\Gamma^{x}_{xt} \\ &= \frac{\left(\partial_t a\right)^2}{a^4-1} - a\partial_t^2 a, \end{aligned}$$

$$R^{t}{}_{yty} = -R^{t}{}_{yyt} = R^{t}{}_{xtx} = -R^{t}{}_{xxt} = -\Gamma^{t}_{xy}\Gamma^{x}_{yt} = \frac{\left(a\partial_t a\right)^2}{a^4-1}$$

$$\begin{aligned} R^{x}{}_{ttx} = -R^{x}{}_{txt} = \partial_t \Gamma^{x}_{tx} + \Gamma^{x}_{\lambda t}\Gamma^{\lambda}_{tx} &= \partial_t \Gamma^{x}_{tx} + \Gamma^{x}_{xt}\Gamma^{x}_{tx} + \Gamma^{x}_{yt}\Gamma^{y}_{tx} \\ &= \frac{a^2}{\left(a^4-1\right)^2}\left[a\partial_t^2 a\left(a^4-1\right) - 2\left(\partial_t a\right)^2\right]. \end{aligned}$$

$$\begin{aligned} R^{y}{}_{tty} = -R^{y}{}_{tyt} = \partial_t \Gamma^{y}_{ty} + \Gamma^{y}_{\lambda t}\Gamma^{\lambda}_{ty} &= \partial_t \Gamma^{y}_{ty} + \Gamma^{y}_{xt}\Gamma^{x}_{ty} + \Gamma^{y}_{yt}\Gamma^{y}_{ty} \\ &= \frac{a^2}{\left(a^4-1\right)^2}\left[a\partial_t^2 a\left(a^4-1\right) - 2\left(\partial_t a\right)^2\right], \end{aligned}$$

$$\begin{aligned} R^{y}{}_{ttx} = R^{x}{}_{tty} &= \partial_t \Gamma^{y}_{tx} + \Gamma^{y}_{\lambda t}\Gamma^{\lambda}_{tx} \\ &= \frac{\left(a^4-1\right)a\partial_t^2 a - \left(1+a^4\right)\left(\partial_t a\right)^2}{\left(a^4-1\right)^2}, \end{aligned}$$

$$\begin{aligned} R^{x}{}_{yxy} = -R^{x}{}_{yyx} &= -\Gamma^{x}_{ty}\Gamma^{t}_{yx} \\ &= \frac{\left(a\partial_t a\right)^2}{a^4-1}, \end{aligned}$$

$$R^{x}{}_{xyx} = -\Gamma^{x}_{\lambda x}\Gamma^{\lambda}_{xy} = -\frac{a^4\partial_t a}{a^4-1}.$$

1.1.5 Ricci tensor

Tracing over the Riemann tensor gives the Ricci tensor,

$$R^{\xi}{}_{\mu\xi\nu} = R_{\mu\nu},$$

with non-trivial components:

$$\begin{aligned}
R_{tt} &= \cancelto{0}{R^{t}{}_{ttt}} + R^{x}{}_{txt} + R^{y}{}_{tyt} + \cancelto{0}{R^{z}{}_{tzt}} \\
&= \frac{2a^2}{\left(a^4-1\right)^2}\left[2\left(\partial_t a\right)^2 - a\partial_t^2 a\left(a^4-1\right)\right], \\
R_{yy} = R_{xx} &= R^{t}{}_{yty} + R^{x}{}_{yxy} = \frac{2\left(a\partial_t a\right)^2}{a^4-1} \\
R_{xy} = R_{yx} &= R^{t}{}_{xty} + R^{x}{}_{xxy} = \frac{\left(a^4-1\right)\left(\partial_t a\right)^2}{a^4-1} + a\partial_t^2 a = \left(\partial_t a\right)^2 + a\partial_t^2 a.
\end{aligned}$$

1.1.6 Ricci scalar

Tracing again over all remaining indices, one obtains from the Ricci tensor the Ricci scalar of curvature,

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} = g^{tt}R_{tt} + g^{xx}R_{xx} + g^{yy}R_{yy} + g^{xy}R_{xy} + g^{yx}R_{yx} \\
&= \frac{2\left(a^4-5\right)\left(a\partial_t a\right)^2 + 4a^3\partial_t^2 a\left(a^4-1\right)}{\left(a^4-1\right)^2}.
\end{aligned}$$

1.2 Weyl tensor

In 4 dimensions, the Weyl tensor is given by

$$\begin{aligned}
C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} &+ \frac{1}{2}\left(g_{\mu\sigma}R_{\nu\lambda} - g_{\mu\lambda}R_{\nu\sigma} + g_{\nu\lambda}R_{\mu\sigma} - g_{\nu\sigma}R_{\mu\lambda}\right) \\
&+ \frac{1}{6}R\left(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}\right).
\end{aligned}$$

Its non-trivial components are

$$
\begin{aligned}
C_{xttx} = C_{txxt} = -C_{txtx} = -C_{xtxt} = C_{ytty} = C_{tyyt} = -C_{tyty} = -C_{ytyt} \\
&= g_{xx} R^{x}{}_{ttx} + g_{xy} R^{y}{}_{ttx} \\
&+ \frac{1}{2}\left(g_{xx} R_{tt} - \cancelto{0}{g_{xt}} R_{tx} + g_{tt} R_{xx} - \cancelto{0}{g_{tx}} R_{xt} \right) \\
&+ \frac{1}{6} R \left(\cancelto{0}{g_{xt} g_{tx}} - g_{xx} g_{tt} \right) \\
&= \frac{a^2 \left(\left(1 + a^4\right) \left(\partial_t a\right)^2 - a \partial_t^2 a \left(a^4 - 1\right) \right)}{3\left(a^4 - 1\right)}
\end{aligned}
$$

$$
\begin{aligned}
C_{yxyx} = C_{xyxy} = -C_{xyyx} = -C_{yxxy} \\
&= g_{yx} \cancelto{0}{R^{x}{}_{xyx}} + g_{yy} R^{x}{}_{yyx} \\
&+ \frac{1}{2}\left(g_{yx} R_{xy} - g_{yy} R_{xx} + g_{xy} R_{yx} - g_{xx} R_{yy} \right) \\
&+ \frac{1}{6} R \left(g_{yy} g_{xx} - g_{yx} g_{xy} \right). \\
&= \frac{1}{3}\left(\left(1 + a^4\right) \frac{\left(a \partial_t a\right)^2}{a^4 - 1} - a^3 \partial_t^2 a \right),
\end{aligned}
$$

$$
\begin{aligned}
C_{txty} = C_{xtyt} = -C_{xtty} = -C_{txyt} = C_{tytx} = C_{ytxt} = -C_{yttx} = -C_{tyxt} \\
&= R_{txty} + \frac{1}{2}\left(\cancelto{0}{g_{ty}} R_{xt} - g_{tt} R_{xy} + \cancelto{0}{g_{xt}} R_{ty} - g_{xy} R_{tt} \right) \\
&+ \frac{1}{6} R \left(g_{tt} g_{xy} - g_{ty} \cancelto{0}{g_{xt}} \right) \\
&= \frac{\left(a^4 - 3\right) \left(a \partial_t^2 a \left(a^4 - 1\right) - \left(\partial_t a\right)^2 \left(1 + a^4\right) \right)}{6\left(a^4 - 1\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
C_{tzzt} = -C_{tztz} = C_{zttz} = -C_{ztzt} \\
&= \cancelto{0}{R_{tzzt}} + \frac{1}{2}\left(g_{tt} \cancelto{0}{R_{zz}} - 2 \cancelto{0}{g_{tz} R_{zt}} + g_{zz} R_{tt} \right) \\
&+ \frac{1}{6} R \left(\cancelto{0}{g_{tz} g_{zt}} - g_{tt} g_{zz} \right) \\
&= \frac{a^2 \left(a \partial_t^2 a \left(a^4 - 1\right) - \left(\partial_t a\right)^2 \left(1 + a^4\right) \right)}{2\left(a^4 - 1\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
C_{xzzx} &= C_{yzzy} = C_{zxxz} = -C_{zxzx} = -C_{xzxz} = C_{zyyz} = -C_{zyzy} = -C_{yzyz} \\
&= \cancelto{0}{R_{xzzx}} + \frac{1}{2}\left(g_{xx}\cancelto{0}{R_{zz}} - 2\cancelto{0}{g_{xz}R_{zx}} + g_{zz}R_{xx}\right) \\
&+ \frac{1}{6}R\left(\cancelto{0}{g_{xz}g_{zx}} - g_{xx}g_{zz}\right) \\
&= \frac{2a^2\left(\left(1+a^4\right)\left(\partial_t a\right)^2 - a\partial_t^2 a\left(a^4-1\right)\right)}{3\left(a^4-1\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
C_{zxyz} &= C_{zyxz} = C_{yzzx} = C_{xzzy} = -C_{xzyz} = -C_{zxzy} = -C_{yzxz} = -C_{zyzx} \\
&= R_{zxyz} + \frac{1}{2}\left(g_{zz}R_{xy} - \cancelto{0}{g_{zy}R_{xz}} + g_{xy}\cancelto{0}{R_{zz}} - \cancelto{0}{g_{xz}R_{zy}}\right)^{0} \\
&+ \frac{1}{6}R\left(\cancelto{0}{g_{zy}g_{xz}} - g_{zz}g_{xy}\right) \\
&= \frac{\left(a^4+3\right)\left(a\partial_t^2 a\left(a^4-1\right) - \left(\partial_t a\right)^2\left(a^4+1\right)\right)}{6\left(a^4-1\right)^2}.
\end{aligned}
$$

1.3 Geodesic equations

For a path $\gamma(\lambda) = (t(\lambda), x(\lambda), y(\lambda), z(\lambda))$, the geodesic equations are given by

$$\frac{d^2\gamma^\mu}{d^2\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{d\gamma^\rho}{d\lambda}\frac{d\gamma^\sigma}{d\lambda} = 0.$$

Recall the Christoffel symbols:

$$
\begin{aligned}
\Gamma^t_{xy} &= \frac{1}{2}g^{tt}\left(-g_{xy,t}\right) = a\partial_t a, \\
\Gamma^x_{xt} &= \Gamma^y_{yt} = \frac{a^3\partial_t a}{a^4-1}, \\
\Gamma^y_{xt} &= \Gamma^x_{yt} = \frac{a\partial_t a}{1-a^4}, \\
\Gamma^z_{\mu\nu} &= 0, \forall\mu,\nu.
\end{aligned}
$$

The explicit geodesic equations are then

$$\begin{aligned}
\frac{d^2t}{d^2\lambda^2} + 2a\partial_t a \frac{dx}{d\lambda}\frac{dy}{d\lambda} &= 0, \\
\frac{d^2x}{d^2\lambda^2} + 2\frac{a\partial_t a}{1-a^4}\frac{dt}{d\lambda}\left(a\frac{dx}{d\lambda} + \frac{dy}{d\lambda}\right) &= 0, \\
\frac{d^2y}{d^2\lambda^2} + 2\frac{a\partial_t a}{1-a^4}\frac{dt}{d\lambda}\left(a\frac{dy}{d\lambda} + \frac{dx}{d\lambda}\right) &= 0, \\
\frac{d^2z}{d^2\lambda^2} &= 0.
\end{aligned}$$