Tr(M^2) :

$$M^{i}_{\ j} = g^{ik}\partial_u g_{kj} - \frac{1}{D-2} g^{mn}\partial_u g_{mn}\cdot\delta_{ij}$$

$$(M^2)_{ij} = \Big(g^{ik}\partial_u g_{kl} - \frac{1}{D-2}\, g\underbrace{^{mn}\partial_u g_{mn}}_{X}\delta_{il}\Big)\cdot\Big(g^{lp}\partial_u g_{pj} - \frac{1}{D-2}\, g\, X\,\delta_{lj}\Big)$$

$$= g^{ik}\partial_u g_{kl}\cdot g^{lp}\partial_u g_{pj} - \frac{1}{D-2} X\cdot g^{ik}\partial_u g_{kl}\cdot\delta_{lj} - \frac{1}{D-2} X\, g^{lp}\partial_u g_{pj}\,\delta_{il}$$

$$+ \frac{1}{(D-2)^2} X^2 \delta_{il}\delta_{lj}$$

$$\Rightarrow \mathrm{Tr}(M^2) = (M^2)^{i}_{\ i} = g^{ik}\partial_u g_{kl}\cdot g^{lp}\partial_u g_{pi} - \frac{1}{D-2} X g^{ik}\partial_u g_{kl}\,\delta_{li} - \frac{1}{D-2} X\, g^{lp}\partial_u g_{pi}\,\delta_{il}$$

$$+ \frac{1}{(D-2)^2} X^2 \delta_{il}\,\delta_{li}$$

$$= \frac{X^2}{D-2} + g^{ik}\partial_u g_{kl}\cdot g^{lp}\partial_u g_{pi} - \frac{1}{D-2} X\, g^{ik}\partial_u g_{ki} - \frac{1}{D-2} X\, g^{lp}\partial_u g_{pi}$$

$$= \left\{ g^{ik}\partial_u g_{ki} = g^{ip}\partial_u g_{pi} = X \right\} = g^{ik}\partial_u g_{kl}\cdot g^{lp}\partial_u g_{pi} - \frac{X^2}{D-2}$$

$$\Rightarrow \frac{1}{4}\mathrm{Tr}(M^2) = \left(\frac{1}{2} g^{ik}\partial_w g_{kl}\right)\cdot\left(\frac{1}{2} g^{lp}\partial_u g_{pi}\right) - \frac{X^2}{4(D-2)}$$

$$R_{uu} = \partial_\lambda \Gamma^\lambda_{uu} - \partial_u \Gamma^\lambda_{u\lambda} + \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{uu} - \Gamma^\lambda_{\rho u}\Gamma^\rho_{u\lambda}$$

$$\Gamma^\lambda_{uu} = 0 \qquad , \Gamma^\lambda_{u\lambda} = \frac{1}{2} g^{\lambda\sigma}\partial_u g_{\lambda\sigma}$$

$$\Gamma^\lambda_{\rho u} = \frac{1}{2} g^{\lambda\sigma}\partial_u g_{\sigma\rho} \Rightarrow \Gamma^\lambda_{\rho u}\Gamma^\rho_{u\lambda} = \left(\frac{1}{2} g^{\lambda\sigma}\partial_u g_{\sigma\rho}\right)\cdot\left(\frac{1}{2} g^{\rho\ell}\partial_u g_{\ell\lambda}\right)$$

$$R_{uu} - \frac{1}{2}\cancel{g_{uu}}R = 8\pi G T_{uu}$$

~~משוואות איינשטיין הופכות ל~~:

~~סך,~~ נקבל את צד ימין של משוואת ריצ'אודורי:

$$\frac{1}{2}\cdot\partial_u\left[g^{\lambda\sigma}\partial_u g_{\lambda\sigma}\right] + \cancel{\left(\frac{1}{2} g^{\lambda\sigma}\partial_u g_{\sigma\rho}\right)\cdot\left(\frac{1}{2} g^{\rho\ell}\partial_u g_{\ell\lambda}\right)} - \cancel{\left(\frac{1}{2} g^{ik}\partial_u g_{k\ell}\right)}\cdot\left(\frac{1}{2} g^{\ell p}\partial_u g_{pi}\right)$$

$$+ \frac{\left(g^{ik}\partial_u g_{ki}\right)^2}{2(D-2)}$$

$$= \left\{X = g^{\lambda\sigma}\partial_u g_{\lambda\sigma}\right\} = \frac{1}{2}\partial_u X + \frac{X^2}{4(D-2)}$$

נניח שמתקיים: $\frac{X}{2} = \theta$

$$\theta = \frac{d}{du}\left(\ln\left(\sqrt{\det g}\right)\right) = \frac{1}{2}\frac{1}{\det g}\cdot\partial_u(\det g)$$

$$X = g^{\lambda\sigma}\partial_u g_{\lambda\sigma} = \left\{g^{\lambda\sigma} = \frac{1}{\det g}\cdot \mathrm{adj}(g)_{\lambda\sigma}\right\} = \frac{1}{\det g}\cdot \mathrm{adj}(g)_{\lambda\sigma}\,\partial_u g_{\lambda\sigma}$$

$$= \frac{1}{\det g}\cdot\partial_u(\det g)$$

$$\Rightarrow \quad \theta = \frac{X}{2} \quad \Rightarrow$$ משוואת ריצ'אודורי מתקיימת

$$\partial_u\theta + \frac{1}{D-2}\theta^2 = -8\pi G T_{uu} - \frac{1}{4}\mathrm{Tr}(M^2)$$

$\blacksquare$

ב) כתבו במפורש את משוואת רייצ'אודורי עבור ג'אודזיות רדיאליות בשוורצשילד

נתון: $$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

$$= \quad -du\,dV + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

כאשר: $$u = u(r,t)$$
$$V = V(r,t)$$

$$M^i_j = g^{ik}\partial_u g_{kj} - \frac{1}{2}g^{mn}\partial_u g_{mn}$$

$$g^{mn}\partial_u g_{mn} = \frac{1}{r^2}\partial_u(r^2) + \frac{1}{r^2\sin^2\theta}\partial_u(r^2\sin^2\theta) = \frac{2}{r^2}\partial_u(r^2) = \frac{4}{r}\left(\frac{\partial r}{\partial u}\right)$$

$$g^{ik}\partial_u g_{kj} = \frac{1}{r^2}\partial_u(r^2)\,\delta_{ij} = \frac{2}{r}\left(\frac{\partial r}{\partial u}\right)\delta_{ij}$$

$$\Rightarrow \quad M^i_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad Tr(M^2) = 0$$

$$\Rightarrow \qquad \partial_u\theta + \frac{1}{2}\theta^2 = -8\pi G T_{uu}$$

$$8\pi G T_{uu} = R_{uu} = -\partial_u\Gamma^{\lambda}_{u\lambda} - \Gamma^{\lambda}_{u\rho}\Gamma^{\rho}_{\lambda u}$$
$$= -\partial_u\left[\frac{1}{2}g^{\lambda\sigma}\partial_u g_{\sigma\lambda}\right] - \frac{1}{4}\left(g^{11}\partial_u g_{11}\right)^2 - \frac{1}{4}\left(g^{22}\partial_u g_{22}\right)^2$$
$$= -\partial_u\left[\frac{1}{2}g^{11}\partial_u g_{11} + \frac{1}{2}g^{22}\partial_u g_{22}\right] - \frac{1}{4}\left(g^{11}\partial_u g_{11}\right)^2 - \frac{1}{4}\left(g^{22}\partial_u g_{22}\right)^2$$
$$= -\partial_u\left[\frac{2}{r}\frac{dr}{du}\right] - \frac{2}{r^2}\left(\frac{dr}{du}\right)^2 = +\frac{2}{r^2}\left(\frac{dr}{du}\right)^2 - \frac{2}{r}\frac{d^2r}{du^2} - \frac{2}{r^2}\left(\frac{dr}{du}\right)^2$$
$$= -\frac{2}{r}\frac{d^2r}{du^2}$$

$$\Rightarrow \qquad \boxed{\partial_u\theta + \frac{1}{2}\theta^2 = \frac{2}{r}\frac{d^2r}{du^2}}$$

$$\partial_u = \frac{d}{du} = \frac{dr}{du}\cdot\frac{\partial}{\partial r} + \frac{dt}{du}\cdot\frac{\partial}{\partial t} \qquad \wedge$$ det g ואי-תלות ב-t det g

$$\Downarrow$$

$$\boxed{\frac{dr}{du}\cdot\frac{d\theta}{dr} + \frac{1}{2}\theta^2 = \frac{2}{r}\frac{d^2r}{du^2}}$$

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$$du = \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} dt + -\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} dr$$ סכמיו נקבל מקומות:

$$dv = \left(1-{}^{2M}/_{r}\right)^{1/2} dt + \left(1-{}^{2M}/_{r}\right)^{-1/2} dr$$

ואכן $$-du\,dv = -\left[\left(1-\frac{2M}{r}\right)^{1/2} dt - \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} dr\right] \cdot \left[\left(1-\frac{2M}{r}\right)^{1/2} dt + \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} dr\right]$$

$$= -\left(1-\frac{2M}{r}\right) dt^2 - \cancel{dt\,dr} + \cancel{dt\,dr} + \left(1-\frac{2M}{r}\right)^{-1} dr^2 = -\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2$$

$$\begin{cases} \frac{1}{2}(du + dv) = \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} dt \Rightarrow \frac{\left(1-\frac{2M}{r}\right)^{-1/2}}{2} \cdot (du + dv) = dt \\ \frac{1}{2}(dv - dv) = \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} dr \Rightarrow \frac{\left(1-\frac{2M}{r}\right)^{1/2}}{2} (dv - du) = dr \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \frac{dr}{du} = -\frac{1}{2}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} \\ \frac{d^2r}{du^2} = \left(\frac{dr}{du}\right)\frac{d}{dr}\left(\frac{dr}{du}\right) = -\frac{1}{2}\cancel{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}} \cdot \frac{2M}{r^2} \cdot \frac{1}{2} \cdot \cancel{\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}} = -\frac{M}{2r^2} \end{cases}$$

נציב ב- $\frac{dr}{du} \cdot \frac{d\theta}{dr} + \frac{1}{2}\theta^2 = \frac{2}{r}\frac{d^2r}{du^2}$:

$\Rightarrow$

$$\boxed{-\frac{1}{2}\left(1-\frac{2M}{r}\right)^{1/2} \cdot d_r\theta + \frac{1}{2}\theta^2 = -\frac{M}{r^3}}$$

$$ds^2 = -dt^2 + a(t)^2 \frac{dr^2}{1-Kr^2} + a(t)^2 r^2 d\Omega^2$$

$$= -du\,dv + g_{ij}\,dx^i dx^j$$

$$M^i_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \Leftarrow \quad g_{ij} = \begin{pmatrix} a(t)^2 r^2 & \\ & a(t)^2 r^2 \sin^2\theta \end{pmatrix} \quad \text{new}$$

$$8\pi G T_{uu} = R_{uu} = -\partial_u \Gamma^{\lambda}_{u\lambda} - \Gamma^{\lambda}_{u\rho}\Gamma^{\rho}_{\lambda u}$$

$$= -\partial_u\left[\frac{1}{2} g^{\lambda 0}\partial_u g_{0\lambda}\right] - \frac{1}{4}\left(g^{11}\partial_u g_{11}\right)^2 - \frac{1}{4}\left(g^{22}\partial_u g_{22}\right)$$

$$= -\partial_u\left[\frac{1}{2}\frac{1}{a^2(t) r^2}\partial_u\left(a^2(t) r^2\right)\right] - \frac{1}{2}\cdot\left(\frac{1}{a^2(t) r^2}\partial_u\left(a^2(t) r^2\right)\right)^2$$

$$\left\{\frac{\partial}{\partial u} = \frac{dt}{du}\frac{\partial}{\partial t} + \frac{dr}{du}\cdot\frac{\partial}{\partial r}\right\} = -\partial_u\left[\frac{1}{a^2(t) r^2}\left(2\dot a(t) a(t) r^2 \cdot \left(\frac{dt}{du}\right) + 2 r a(t)^2 \cdot \left(\frac{dr}{du}\right)\right)\right]$$

$$- \frac{1}{2}\left[\frac{1}{a^2(t) r^2}\cdot\left(2\dot a(t) a(t) r^2 \left(\frac{dt}{du}\right) + 2 r a(t)^2 \left(\frac{dr}{du}\right)\right)\right]^2$$

$$= -\partial_u\left[2\frac{\dot a(t)}{a(t)}\cdot\left(\frac{dt}{du}\right) + \frac{2}{r}\left(\frac{dr}{du}\right)\right] - \frac{1}{2}\left[4\left(\frac{\dot a(t)}{a(t)}\right)^2\cdot\left(\frac{dt}{du}\right)^2 + \frac{4}{r}\frac{\dot a(t)}{a(t)}\cdot\left(\frac{dr}{du}\right)\left(\frac{dt}{du}\right) + \frac{4}{r^2}\cdot\left(\frac{dr}{du}\right)^2\right]$$

$$= -2\left(\frac{\ddot a(t)}{a(t)}\right)\cdot\left(\frac{dt}{du}\right)^2 - 2\left(\frac{\dot a(t)}{a(t)}\right)\cdot\frac{d^2 t}{du^2} - \frac{2}{r}\left(\frac{d^2 r}{du^2}\right) - \frac{2}{r}\frac{\dot a(t)}{a(t)}\left(\frac{dr}{du}\right)\left(\frac{dt}{du}\right)$$

$$\Rightarrow \boxed{\left(\frac{dt}{du}\right)\cdot\partial_t\Theta + \left(\frac{dr}{du}\right)\partial_r\Theta + \frac{1}{2}\Theta^2 = 2\left(\frac{\ddot a(t)}{a(t)}\right)\cdot\left(\frac{dt}{du}\right)^2 + 2\left(\frac{\dot a(t)}{a(t)}\right)\frac{d^2 t}{du^2} + \frac{2}{r}\left(\frac{d^2 r}{du^2}\right) - \frac{2}{r}\cdot\frac{\dot a(t)}{a(t)}\cdot\left(\frac{dr}{du}\right)\left(\frac{dt}{du}\right)}$$

$$du = dt + \frac{a(t)}{\sqrt{1-Kr^2}}dr \qquad\qquad dt = \frac{1}{2}du + \frac{1}{2}dv$$

$$dv = dt - \frac{a(t)}{\sqrt{1-Kr^2}}dr \quad \Rightarrow \quad dr = \frac{\sqrt{1-Kr^2}}{2a(t)}(du - dv)$$

$$\frac{d^2 r}{du^2} = \frac{dr}{du}\cdot\frac{d}{dr}\left(\frac{\sqrt{1-Kr^2}}{2a(t)}\right) + \frac{dt}{du}\cdot\frac{d}{dt}\left(\frac{\sqrt{1-Kr^2}}{2a(t)}\right) = -\frac{Kr}{4a^2(t)} - \frac{\sqrt{1-Kr^2}\cdot\dot a(t)}{4a^2(t)}$$

$\Rightarrow$

$$\Rightarrow \boxed{\frac{1}{2}\partial_t\Theta + \frac{\sqrt{1-Kr^2}}{2a(t)}\partial_r\Theta + \frac{1}{2}\Theta^2 = \frac{1}{2}\left(\frac{\ddot a(t)}{a(t)}\right) - \frac{K}{2a^2(t)} - \frac{\sqrt{1-Kr^2}\,\dot a(t)}{a^2(t)\, r}} \; - \frac{1}{r}\cancel{\frac{\dot a(t)}{a(t)}\frac{\sqrt{1-Kr^2}}{2a(t)}}$$

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