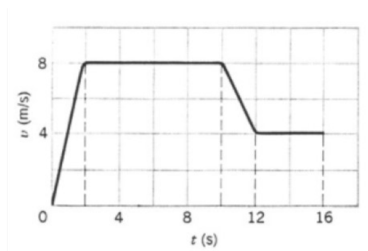
Tutorial 1

1 High School reminder - Runner's velocity

The velocity of a runner during his training is described in the graph below:



1. What is the total distance that the runner runs during his training?
2. Find the expression for $x(t)$ (the runner's position function of t) and sketch it.
3. Find the average velocity $\overline{v}$ of the *runner* during the first 10 seconds.
4. Schematically, sketch a graph describing the acceleration $a(t)$ vs. t.

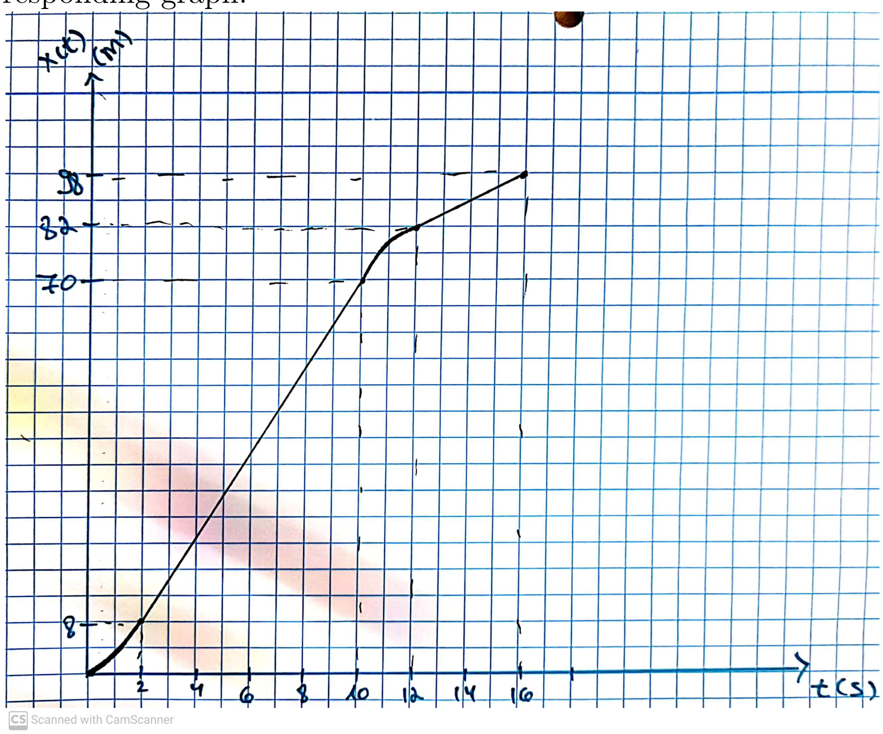
Solution:

1. Let us divide the runner's motion into four stages, for each stage we will solve $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$. Note that a linear slope in $v\,(t)$ signifies constant acceleration:
   $0 - 2\,sec$:
   $$x(t) = 0 + 0 \times t + \frac{1}{2}4t^2 = 2t^2$$
   At $t = 2\,sec$ the runner passed $x(t = 2) = 8\,meters$.
   $2 - 10\,sec$:
   $$x(t) = 8 + 8(t-2) + \frac{1}{2}0(t-2)^2 = 8t - 8$$
   The runner passed $x(t = 10) = 72\,meter$ by the end of the 2^{nd} stage.
   $10 - 12\,sec$:
   $$x(t) = 72 + 8(t-10) + \frac{1}{2}(-2)(t-10)^2 = -t^2 + 28t - 108$$
   $x(t = 12) = 84\,meters$ after the 3^{rd} stage.
   $12 - 16\,sec$:
   $$x(t) = 84 + 4(t-12) = 4t + 34$$
   $x(16) = 100\,meters$ it's the total distance.

2. We solved in 1. :

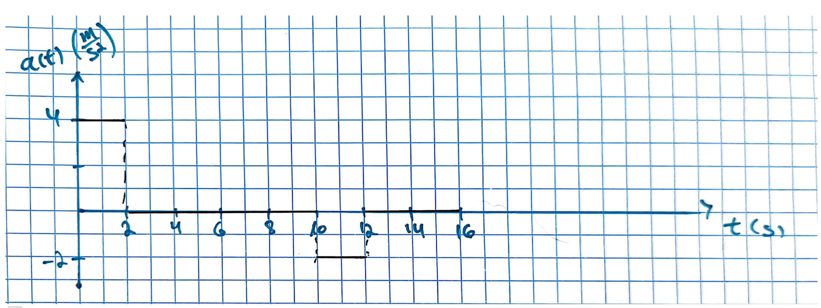
$$x(t) = \begin{cases} t^2 & 0 \leq t \leq 2 \\ 8t - 12 & 2 \leq t \leq 10 \\ -t^2 + 28t - 112 & 10 \leq t \leq 12 \\ 4t + 32 & 12 \leq t \leq 16 \end{cases}$$

The corresponding graph:



3. The average velocity for the first two seconds is $\frac{8+0}{2} = 4$, then we got constant velocity for eight seconds.

$$\overline{v} = \frac{4 \cdot 2 + 8 \cdot 8}{10} = 7.2 \frac{meter}{sec}.$$



4.

2 Substracting vectors

A ball undergoes a displacement $\Delta \boldsymbol{r} = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}$, ending with the position vector $\boldsymbol{r} = 3\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$, given in meters. What was the ball's initial position vector $\boldsymbol{r}_i$?

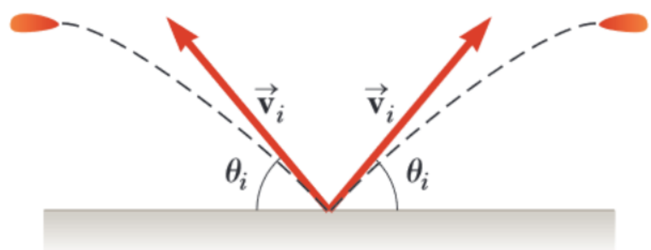
Solution:

A displacement vector is defined as $\Delta \boldsymbol{A} \equiv \boldsymbol{A}_f - \boldsymbol{A}_i$, where i, f subscripts are for the final and initial states. Therefore

$$\begin{aligned} \boldsymbol{r}_i &= \boldsymbol{r} - \Delta \boldsymbol{r} \\ &= (0-2)\,\hat{\boldsymbol{\imath}} + (3-(-3))\,\hat{\boldsymbol{\jmath}} + (-4-6)\,\hat{\boldsymbol{k}} \\ &= -2\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}}. \end{aligned}$$

3 Two Drops

Two drops are splash simultaneously at $t = 0$ as shown in the figure. Find the distance between the drops as a function of time.



Solution:

Since both drops go through the same vertical motion (both share the same initial vertical velocity $|\boldsymbol{v}_i| \sin\theta_i$), the distance between them will be horizontal alone - which can simplify the calculations. Let us solve the problem twice: first, by going through the whole vector calculation, then by simplifying the problem to one axis.

Method 1:
The distance between the two drops can be described by the norm of the difference between their position vectors

$$|\Delta \boldsymbol{r}| = |\boldsymbol{r}_1 - \boldsymbol{r}_2|\,.$$

In order to find $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ we begin with the acceleration vectors,

$$\boldsymbol{a}_1 = \boldsymbol{a}_2 = -g\hat{\boldsymbol{y}},$$

due to gravity alone. Next we integrate the acceleration to find the velocity vectors,

$$\boldsymbol{v} = \int \boldsymbol{a} dt = -gt\hat{\boldsymbol{y}} + \boldsymbol{v}_0,$$

where $\boldsymbol{v}_0$ is determined by the initial conditions (the value of $\boldsymbol{v}$ at $t = 0$):

$$\begin{aligned} \boldsymbol{v}_1 &= -gt\hat{\boldsymbol{y}} + v_i \left(\cos\theta_i \hat{\boldsymbol{x}} + \sin\theta_i \hat{\boldsymbol{y}}\right) \\ \boldsymbol{v}_2 &= -gt\hat{\boldsymbol{y}} + v_i \left(-\cos\theta_i \hat{\boldsymbol{x}} + \sin\theta_i \hat{\boldsymbol{y}}\right). \end{aligned}$$

Integrating again would give us the position vectors,

$$\boldsymbol{r} = \int \boldsymbol{v} dt = -\frac{g}{2}t^2 + \boldsymbol{v}_0 t + \boldsymbol{r}_0,$$

where $\boldsymbol{r}_0$ is the initial position vector (the value of $\boldsymbol{r}$ at $t = 0$), which is zero if we define the origin to be the initial position of the drops:

$$\begin{aligned} \boldsymbol{r}_1 &= \left(v_i \sin\theta_i\, t - \frac{g}{2}t^2\right)\hat{\boldsymbol{y}} + \cos\theta_i\, t\,\hat{\boldsymbol{x}} \\ \boldsymbol{r}_2 &= \left(v_i \sin\theta_i\, t - \frac{g}{2}t^2\right)\hat{\boldsymbol{y}} - \cos\theta_i\, t\,\hat{\boldsymbol{x}}. \end{aligned}$$

Thus the distance between the drops is

$$\begin{aligned} |\Delta\boldsymbol{r}| &= \left|\left(v_i \sin\theta_i\, t - \frac{g}{2}t^2\right)\hat{\boldsymbol{y}} + \cos\theta_i\, t\,\hat{\boldsymbol{x}} - \left[\left(v_i \sin\theta_i\, t - \frac{g}{2}t^2\right)\hat{\boldsymbol{y}} - \cos\theta_i\, t\,\hat{\boldsymbol{x}}\right]\right| \\ &= |2\cos\theta_i\, t\,\hat{\boldsymbol{x}}| \\ &= 2\cos\theta_i\, t, \end{aligned}$$

we've found that the $\hat{\boldsymbol{y}}$ components cancelled and the distance was determined by the $\hat{\boldsymbol{x}}$ components alone.

Method 2:

Since we see that the distance between the drops is horizontal we can completely ignore the y axis (of course we need to take it into account if we want to find when the drops get back to the initial vertical position) and calculate the horizontal distance. Since the motion is antisymmetric in the x axis, i.e. one drop moves in one direction and the second one in the other, we may calculate the distance of one drop from the origin and then double it to find the total distance between the drops. The acceleration in the x direction is zero, thus the velocity is constant

$$v_x = v_i \cos\theta.$$

The position of the drop is found by integrating over the velocity

$$x = \int v_x dt = \int v_i \cos\theta\, dt = v_i \cos\theta\, t + \cancelto{0}{x_0}.$$

Therefore the distance between the drops, which is twice the distance of one drop from the origin, is

$$\Delta x = 2x = 2v_i \cos\theta\, t,$$

which is the same result we would found in the longer first method.

4 3D Time Dependent Acceleration:

A particle's motion is given by $\overrightarrow{a(t)} = 12t^2\hat{\imath} + (18t-8)\hat{\jmath} - 6t\hat{k}$, with the initial conditions of $\overrightarrow{r(t=0)} = (3,-1,4)$ and $\overrightarrow{v(t=0)} = (6,5,-8)$.
Find:

1. $\overrightarrow{v(t)}$
2. $\overrightarrow{r(t)}$

Solution:

1. The velocity vector is obtained by integrating over the acceleration vector:

$$\begin{aligned}\overrightarrow{v(t)} &= \int 12t^2\,dt\,\hat{\imath} + \int 18t - 8\,dt\,\hat{\jmath}\; + \int -6t\,dt\,\hat{k} = \\ &= (4t^3 + C_i)\hat{\imath} + (9t^2 - 8t + C_j)\hat{\jmath} + (-3t^2 + C_k)\hat{k}\end{aligned}$$

Since we do not specify boundaries for the integration we get 3 additional constants, which we find by considering the initial conditions. We set $t = 0$ and solve:

$$6 = 4\,0^3 + C_i \Longrightarrow C_i = 6$$

$$5 = 9\,0^2 - 8\,0 + C_j \Longrightarrow C_j = 5$$

$$-8 = -3\,0^2 + C_k \Longrightarrow C_k = -8$$

We find the velocity to be $\overrightarrow{v(t)} = (4t^3+6)\hat{\imath} + (9t^2-8t+5)\hat{\jmath} + (-3t^2-8)\hat{k}$.
2.In the same manner, we obtain the position vector by integrating over the velocity vector:

$$\begin{aligned}\overrightarrow{r(t)} &= \int 4t^3 + 6\,dt\,\hat{\imath} + \int 9t^2 - 8t + 5\,dt\,\hat{\jmath} + \int -3t^2 - 8\,dt\,\hat{k} = \\ &= (t^4 + 6t + D_i)\hat{\imath} + (3t^3 - 4t^2 + 5t + D_j)\hat{\jmath} + (-t^3 - 8t + D_k)\hat{k}\end{aligned}$$

We set $t = 0$ again, to find

$$D_i = 3, \qquad D_j = -1, \qquad D_k = 4,$$

thus the position vector is $\overrightarrow{r(t)} = (t^4+6t+3)\hat{\imath} + (3t^3-4t^2+5t-1)\hat{\jmath} + (-3t^2-8t+4)\hat{k}$.