Tirgul 3

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1 Uniform Circular Motion

A particle moves in the $x - y$ plane in a circular counterclockwise motion, according to

$$\overrightarrow{r(t)} = R\left(\cos(\omega t), \sin(\omega t)\right).$$

1. Where will the particle be at $t = 0$? When will the particle return there for the first time?
2. Find $\overrightarrow{v(t)}$ and calculate its magnitude, $\left|\overrightarrow{v(t)}\right|$.
3. Calculate the scalar product $\overrightarrow{r(t)} \cdot \overrightarrow{v(t)}$.
4. Can you guess the direction of the acceleration? Find $\overrightarrow{a(t)}$ and see if you were right.
5. Repeat 2-4 using polar coordinates.

Solution:

1. Let us set $t = 0$ in $\overrightarrow{r(t)}$, $\cos(0) = 1$ and $\sin(0) = 0$, thus we get $\overrightarrow{r(t=0)} = (R, 0)$.
In order to find when the particle returns we need to find t that meets the following conditions:

$$1.\cos(\omega t) = 1$$

$$2.\sin(\omega t) = 0$$

We deduce these conditions for t:

$$1.\omega t = 0 + 2\pi\, k$$

$$2.\omega t = 0 + \pi\, k$$

Where k is an integer. The first time we meet both conditions simultaneously after $t = 0$ is at $t = \frac{2\pi}{\omega}$.
2.

$$\overrightarrow{v(t)} = \frac{d\overrightarrow{r(t)}}{dt}$$

Use the Chain Rule to differentiate the trigonometric functions:

$$\frac{d}{dt}\left(f(\omega t)\right) = \frac{d(\omega t)}{dt}\frac{d}{d(\omega t)}\left(f(\omega t)\right)$$

We get $\overrightarrow{v(t)} = R\omega\left(-\sin(\omega t), \cos(\omega t)\right)$.

The magnitude is:

$$\sqrt{\overrightarrow{v(t)} \cdot \overrightarrow{v(t)}} = \sqrt{R^2\omega^2\left((-\sin(\omega t))(-\sin(\omega t)) + \cos(\omega t)\cos(\omega t)\right)} = R\omega\sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = R\omega$$

so we get $\left|\overrightarrow{v(t)}\right| = R\omega$.
Notice! Although the velocity vector is time dependent its magnitude is independent of time, try to think why it is so.
3.

$$\overrightarrow{r(t)} \cdot \overrightarrow{v(t)} = R^2\omega\left(-\sin(\omega t)\cos(\omega t) + \sin(\omega t)\cos(\omega t)\right) = 0$$

$\overrightarrow{r(t)}$ and $\overrightarrow{v(t)}$ are orthogonal vectors as you would expect in a circular motion.
4.

$$\overrightarrow{a(t)} = \frac{d\overrightarrow{v(t)}}{dt} = R\omega^2\left(-\cos(\omega t), -\sin(\omega t)\right)$$

The direction of $\overrightarrow{a(t)}$ is opposite to the direction of $\overrightarrow{r(t)}$.

5. In polar coordinates $\rho = R$, $\phi = \omega t$ and $\overrightarrow{r(t)} = R\hat{r}$.
Calculating the time derivative of $\overrightarrow{r(t)}$:

$$\frac{d}{dt}(R\hat{r}) = \frac{dR}{dt}\hat{r} + R\frac{d\hat{r}}{dt}$$

R is not changing with time.
$\hat{\mathrm{r}} = \cos\phi\hat{i} + \sin\phi\hat{j}$, so $\frac{d\hat{r}}{dt} = \left(-\sin\phi\hat{i} + \cos\phi\hat{j}\right)\frac{d\phi}{dt} = \omega\hat{\phi}$. and

$$\overrightarrow{v(t)} = R\omega\hat{\phi}.$$

$\hat{\phi}\cdot\hat{\phi} = 1$ and we could easily find the magnitude $\left|\overrightarrow{v(t)}\right| = R\omega$.
Since $\hat{r}\cdot\hat{\phi} = 0$ it is clear that $\overrightarrow{r(t)} \cdot \overrightarrow{v(t)} = 0$.
As before we can take a time derivative of the base vector $\hat{\phi}$: $\frac{d\hat{\phi}}{dt} = -\omega\hat{r}$.
We get the acceleration $\overrightarrow{a(t)} = -R\omega^2\hat{r}$ in an opposite direction to $\overrightarrow{r(t)}$.

2 Accelerations in Circular Motion

A particle at rest with constant tangential acceleration a_t starts to move along a cirlce of radius R.
Find the radial acceleration a_r:

a. As a function of time.

b. As a function of the angle of rotation θ.

Solution:

Lets take a look at the second derivative of the location vector in polar coordinates with respect to time:

$$\boldsymbol{a}(t) = \frac{d^2\boldsymbol{r}(t)}{dt^2} = \frac{d^2\left(R\hat{\boldsymbol{r}}\right)}{dt^2} = \frac{d}{dt}\left[\frac{dR}{dt}\hat{\boldsymbol{r}} + R\frac{d\hat{\boldsymbol{r}}}{dt}\right] = \left\{\begin{array}{c}\text{R is a constant on a circle}\\ \text{and } \dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}\end{array}\right\} = R\frac{d}{dt}\left[\dot{\theta}\hat{\boldsymbol{\theta}}\right] =$$
$$= R\left[\frac{d\dot{\theta}}{dt}\hat{\boldsymbol{\theta}} + \dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}\right] = \left\{\begin{array}{c}\text{R is a constant on a circle}\\ \text{and } \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}\end{array}\right\} = R\ddot{\theta}\hat{\boldsymbol{\theta}} + R\dot{\theta}^2(-\hat{\boldsymbol{r}})$$

we can see that the tangential acceleration (given by $\boldsymbol{a}\cdot\hat{\boldsymbol{\theta}}$) is $a_t = R\ddot{\theta}$ and the radial acceleration (given by $\boldsymbol{a}\cdot\hat{\boldsymbol{r}}$) is $a_r = R\dot{\theta}^2$.
Notice that we can directly use the connection between the magnitude of the radial acceleration a_r and the magnitude of the tangential velocity v in a ciruclar motion, given by: $a_r = \frac{v^2}{R} = \frac{(R\dot{\theta})^2}{R} = R\dot{\theta}^2$.
Therefor we can find the second derivation by time of the angle of rotation θ:

$$\ddot{\theta} = \frac{a_t}{R}$$

Because $\boldsymbol{a_t}$ is constant

$$\dot{\theta}(t) = \frac{a_t}{R}t + C$$

and we know that for $t = 0$ the velocity is zero. It means that $R\dot{\theta} = 0 \Rightarrow \dot{\theta}(t = 0) = 0$. So C vanished.
1. For $a_r = a_r(t)$ we simply set the expression for $\dot{\theta}(t)$ and we get:

$$a_r(t) = R\left(\frac{a_t}{R}t\right)^2 = \frac{a_t^2}{R}t^2$$

2. For $a_r = a_r(\theta)$: Let us find θ as a function of t by integrating by time the expression we found for $\dot{\theta}$:

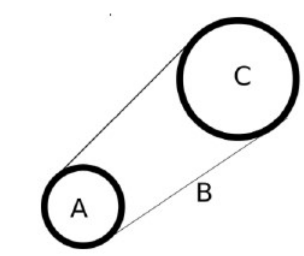
$$\int \dot{\theta}(t)dt = \int \frac{a_t}{R}t\,dt \Rightarrow \theta(t) = \frac{1}{2}\frac{a_t}{R}t^2$$

Here we had the freedom to choose a coordinate system in which $\theta(t = 0) = 0$.
Let's replace $t^2 \to a_r\frac{R}{a_t^2}$ and we get:

$$a_r(\theta) = 2a_t\theta.$$

3 Two Wheels and a Belt

A certain machine contains two wheels connected by a tensioned belt (B), wheel A of radius 10 centimeters and wheel C of radius 25 centimeters.
Both wheels are at rest until wheel A start to rotate and increases its angular velocity at a steady rate of $1.60\frac{rad}{sec^2}$.
How long will it take for wheel C to reach 100 rotations per minute?
Assume that the belt does not slip.



Solution:

We denote v for the tangential velocity of the belt, and for the angular velocities of the wheels ω_A and ω_C.
We can describe the connection between the angular velocity of each wheel to v by:

$$\omega_i r_i = v$$

so the relation between ω_A and ω_C is given by:

$$\omega_A r_A = \omega_C r_C.$$

If wheel C is rotating 100 times per minute then we can calculate the velocity of the belt at that time to be:

$$v_f = \frac{\Delta l}{\Delta t} = \frac{100 \cdot 2\pi\,(0.25m)}{60sec} \backsimeq 2.62\frac{m}{sec}$$

Wheel C's angular velocity at this time is $\omega_{C_f} = \frac{v_f}{r_C} \backsimeq 10.47\frac{rad}{sec}$.
We know wheel A's angular acceleration to be

$$\alpha_A = 1.6\frac{rad}{sec^2}$$

so

$$\omega_A = \alpha_A t$$

with no additive constant because the wheels start at rest.
All that is left to do is to find t according to the equation:

$$r_A \alpha_A t = \omega_{C_f} r_C$$

$$t \backsimeq 16.36\, sec$$

4 Two Bodies Circular motion

Two bodies move with a simple circular motion in a counterclockwise direction around the origin. The first body moves in a trajectory with radius R and angular velocity ω. The second body moves with radius $4R$. Given that both bodies have the same acceleration,

1. Find the angular velocity of the second body.
2. Find the norm of the distance between the bodies as a function of time, if at $t = 0$ both bodies were on the positive part of the x axis.
3. Find the maximal and minimal distances between the bodies, and the corresponding times.

Solution:

1. For simple circular motion the acceleration can be written in term of angular velocity as

$$a = \omega^2 R,$$

therefore, since the second body has the same acceleration, we find

$$\omega^2 R = \omega'^2 4R \quad \rightarrow \quad \omega' = \frac{\omega}{2}.$$

2. The position vector of an object moving in simple circular motion around the origin can be written generally as

$$\boldsymbol{r}(t) = R\left[\cos(\omega t + \phi)\,\hat{\boldsymbol{\imath}} + \sin(\omega t + \phi)\,\hat{\boldsymbol{\jmath}}\right],$$

where ϕ is determined by the initial conditions - in our case $\phi = 0$. Thus the distance vector, between the two bodies, is

$$\begin{aligned}
\Delta\boldsymbol{r} &= \boldsymbol{r}' - \boldsymbol{r} \\
&= 4R\left[\cos\left(\frac{\omega}{2}t\right)\hat{\boldsymbol{\imath}} + \sin\left(\frac{\omega}{2}t\right)\hat{\boldsymbol{\jmath}}\right] - R\left[\cos(\omega t)\,\hat{\boldsymbol{\imath}} + \sin(\omega t)\,\hat{\boldsymbol{\jmath}}\right] \\
&= R\left\{\left[4\cos\left(\frac{\omega}{2}t\right) - \cos(\omega t)\right]\hat{\boldsymbol{\imath}} + \left[4\sin\left(\frac{\omega}{2}t\right) - \sin(\omega t)\right]\hat{\boldsymbol{\jmath}}\right\}.
\end{aligned}$$

Taking the norm of this vector yields

$$|\Delta\boldsymbol{r}| = R\sqrt{\left[4\cos\left(\frac{\omega}{2}t\right) - \cos(\omega t)\right]^2 + \left[4\sin\left(\frac{\omega}{2}t\right) - \sin(\omega t)\right]^2},$$

it is easy to see that the squared terms, such as $4^2\cos^2\left(\frac{\omega}{2}t\right) + 4^2\sin^2\left(\frac{\omega}{2}t\right)$, are reduces to unity times a factor. The reduced expression takes the form

$$\begin{aligned}
|\Delta\boldsymbol{r}| &= R\sqrt{\left[16\cos^2\left(\frac{\omega}{2}t\right) - 8\cos(\omega t)\cos\left(\frac{\omega}{2}t\right) + \cos^2(\omega t)\right] + \left[16\sin^2\left(\frac{\omega}{2}t\right) - 8\sin(\omega t)\sin\left(\frac{\omega}{2}t\right) + \sin^2(\omega t)\right]} \\
&= R\sqrt{17 - 8\cos(\omega t)\cos\left(\frac{\omega}{2}t\right) - 8\sin(\omega t)\sin\left(\frac{\omega}{2}t\right)},
\end{aligned}$$

using the trigonometric relations

$$\cos(\omega t)\cos\left(\frac{\omega}{2}t\right) = \frac{1}{2}\left[\cos\left(\frac{\omega}{2}t\right) + \cos\left(3\frac{\omega}{2}t\right)\right],$$
$$\sin(\omega t)\sin\left(\frac{\omega}{2}t\right) = \frac{1}{2}\left[\cos\left(\frac{\omega}{2}t\right) - \cos\left(3\frac{\omega}{2}t\right)\right],$$

we find

$$|\Delta \boldsymbol{r}(t)| = R\sqrt{17 - 8\cos\left(\frac{\omega}{2}t\right)}.$$

A short sanity check for $t = 0$ yields $|\Delta \boldsymbol{r}| = 3R$, which is correct.

3. We look for the extremums of the function $|\Delta \boldsymbol{r}(t)|$. It is easy to see that the maximum and minimum values are found for $t = 2\pi n/\omega$, where n is an integer - even for the minimal distance and odd for the maximal, corresponding to $|\Delta \boldsymbol{r}| = 3R$ and $|\Delta \boldsymbol{r}| = 5R$ respectively. This can also be found explicitly by taking the derivative and comparing to zero reads

$$\frac{d}{dt}|\Delta \boldsymbol{r}(t)| = \frac{4\omega R\sin\left(\frac{\omega}{2}t\right)}{2\sqrt{17 - 8\cos\left(\frac{\omega}{2}t\right)}} = 0 \quad \rightarrow \quad \sin\left(\frac{\omega}{2}t\right) = 0,$$

which gives us $t = 2\pi n/\omega$, where n is a integer.