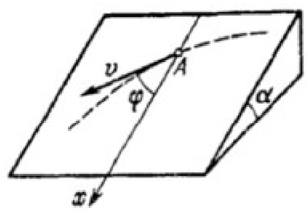
Tutorial 5

1 Sliding on a slope

A small body is put on a slope with the angle α and given an initial velocity v_0.

Initially the angle between the velocity and the fastest descent on the slope is φ.
The friction coefficient is $\mu = \tan\alpha$.
Find the velocity magnitude for $t \to \infty$.



Solution:

Our coordinate system: the x - axis directed downwards with the slope as shown in the figure, the y - axis is pointing to the left, and the z -axis is vertical to the slope and directed from the slope up.
The initial velocity vector in this coordinate system is

$$\boldsymbol{v_0} = v_0\,(\cos\varphi, \sin\varphi, 0)$$

In order to calculate the changes in the velocity we need to find the body acceleration $\boldsymbol{a}$.
Let's write the forces on the body and use Newton's second law:
The gravitational force:

$$\boldsymbol{F_g} = mg\,(\sin\alpha, 0, -\cos\alpha)$$

The normal force due to contact with the slope:

$$\boldsymbol{N} = (0, 0, N)$$

We can find the magnitude N using the second law in the z axis, where $a_z = 0$ because the body stays in contact:

$$N - mg\cos\alpha = 0 \Rightarrow N = mg\cos\alpha$$

and this will stay true as long the body is on the slope (its whole motion).
The last force acting on the body is the friction $\boldsymbol{f}$ that opposing the motion of the body and therefore-

$$\hat{\boldsymbol{f}} = -\hat{\boldsymbol{v}}$$

where $\hat{\boldsymbol{v}}$ is the unit vector in the direction of velocity, and while the body is moving (and not static) $\boldsymbol{f}$ has its maximum value μN.
So

$$\boldsymbol{f} = -\mu mg\cos\alpha\hat{\boldsymbol{v}} = -mg\sin\alpha\hat{\boldsymbol{v}}$$

where the last equality obtained by using $\mu = \tan\alpha$.
The equation of motion in the sloped plane:

$$m\frac{d\boldsymbol{v}}{dt} = mg\sin\alpha\hat{\boldsymbol{x}} - mg\sin\alpha\hat{\boldsymbol{v}} = mg\sin\alpha\left(\hat{\boldsymbol{x}} - \hat{\boldsymbol{v}}\right)$$

We see that the velocity becomes constant when $\hat{\boldsymbol{v}} = \hat{\boldsymbol{x}}$.
Taking scalar products of the equation with $\hat{\boldsymbol{x}}$and $\hat{\boldsymbol{v}}$one gets

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\hat{\boldsymbol{x}}\right) = g\sin\alpha\left(1 - cos\varphi\right)$$

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\hat{\boldsymbol{v}}\right) = \frac{d}{dt}v = -g\sin\alpha\left(1 - cos\varphi\right)$$

Summing up one has

$$\frac{d}{dt}\left(v + \boldsymbol{v}\cdot\hat{\boldsymbol{x}}\right) = 0$$

so

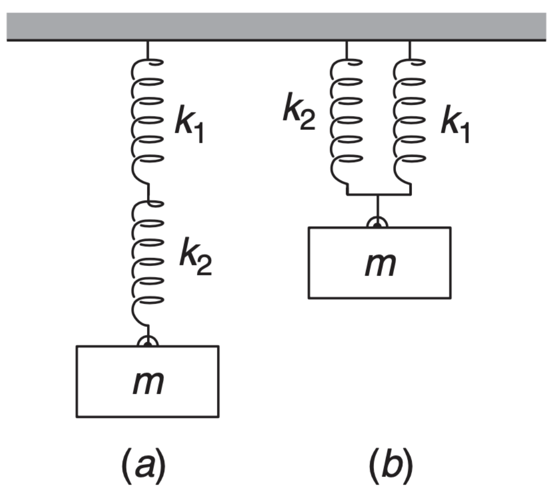
$$v + \boldsymbol{v}\cdot\hat{\boldsymbol{x}} = const$$

In the beginning $v(0) + \boldsymbol{v}(0)\cdot\hat{\boldsymbol{x}} = v_0\left(1 + \cos\varphi\right)$.
In the end $(t \to \infty)$ one has $\boldsymbol{v}\cdot\hat{\boldsymbol{x}} = 1$ and therefore

$$2v(t\to\infty) = v_0\left(1 + \cos\varphi\right) \Rightarrow v(t\to\infty) = \frac{1}{2}v_0\left(1 + \cos\varphi\right).$$

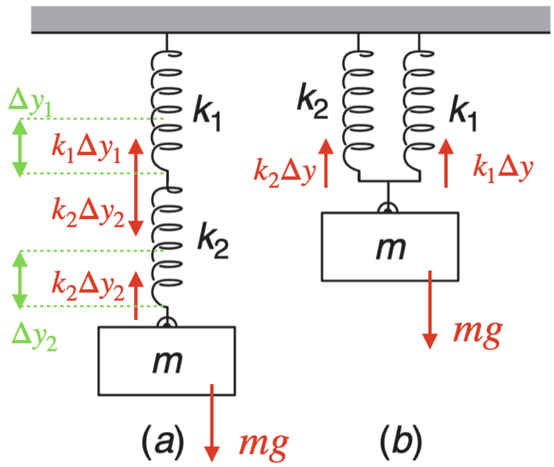
2 Mass and Springs

Find the frequency of oscillation of mass m suspended by two springs having constants k_1 and k_2, in each of the configurations shown.



Solution:

Let us first mark all the forces relevant for the problem:



In configuration (a) the forces equation for point between the two springs reads

$$k_1 \Delta y_1 - k_2 \Delta y_2 = 0,$$

where $\Delta y_1 + \Delta y_2 = \Delta y$, therefore

$$k_1 (\Delta y - \Delta y_2) - k_2 \Delta y_2 = 0 \quad \rightarrow \quad \Delta y_2 = \Delta y \frac{k_1}{k_1 + k_2}.$$

The equation for the mass is

$$k_2 \Delta y_2 - mg = ma,$$

using the expression for Δy_2 we find

$$\Delta y \underbrace{\frac{k_1 k_2}{k_1 + k_2}}_{k} - mg = ma.$$

At equilibrium

$$\Delta y_0 = \frac{mg}{k},$$

therefore, to simplify things, let us choose the origin at Δy_0 so that

$$y \equiv \Delta y - \Delta y_0 \quad \rightarrow \quad \begin{array}{c} \dot{y} = \dot{\Delta y} = v \\ \ddot{y} = \ddot{\Delta y} = a \end{array},$$

and the equation simplifies to be

$$\ddot{y} - \frac{k}{m} y = 0.$$

Solving for y yields

$$y \sim \cos(\omega t),$$

and when plugging the solution into the equation we find

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 k_2}{m (k_1 + k_2)}}.$$

I configuration (b) the forces equation for the mass is

$$\Delta y \left(\underbrace{k_1 + k_2}_{k} \right) - mg = ma.$$

Again, we may write this equation in a simple form, shifting the origin, as

$$\ddot{y} - \frac{k}{m} y = 0,$$

which corresponds to

$$\omega = \sqrt{\frac{k_1 + k_2}{m}}.$$

3 Spinning Room

A man of mass $M = 75\,kg$ is standing in a cylindrical room, which is rotating with an angular speed of ω.

The man stands close to the wall so as not to strain to resist the centrifugal force.
The room has a radius of $R = 1.65\,m$ and its wall's friction coefficient is $\mu = 0.5$.
1. What is the minimum angular velocity ω_{min} at which the person can detach his feet from the floor without slipping to the floor?
2. For some $\omega < \omega_{min}$, how long it will take for the man to slide a distance d towards the floor?

Solution:

1. In the rotating frame of reference of the room, the man will feel a centrifugal force $\overrightarrow{F_c}$ towards the walls and a gravitational force $\overrightarrow{F_g}$ towards earth.
In addition, without friction, the man will slide down. So the wall will exert a frictional force $\overrightarrow{f}$ on the man which directed up.
Working with cylindrical coordinates, the equations of motions are:
In the radial direction, $a_r = 0$ because the man is in contact with the wall:

$$F_c - N = 0 \Rightarrow N = F_c = M\omega^2 R$$

In the vertical direction

$$f - F_g = Ma_z$$

If the man does not slip $a_z = 0$.
And the minimal value for the angular velocity is obtained when the frictional force needs to be with its maximal value $f = \mu N$.

$$\mu M \omega_{min}^2 R = Mg$$

$$\omega_{min}^2 = \frac{g}{\mu R}$$

Setting the numbers in and we get $\omega_{min} = 3.48\,sec^{-1}$.
2.For $\omega < \omega_{min}$ the equation of motion in the vertical axis is

$$M\omega^2 R - Mg = Ma_z$$

$$a_z = \omega^2 R - g$$

and a_z is a negative constant.

$$\Delta z = v_{z,0} t - \frac{1}{2}\left(\omega^2 R - g\right) t^2$$

Initially when the man detached his feet he has no vertical velocity, so

$$v_{z,0} = 0$$

Setting $\Delta z = -d$ one gets

$$t = \sqrt{\frac{2d}{\omega^2 R - g}}.$$

4 Retarding force

A particle of mass m moving along a straight line is acted on by a retarding force (one always directed against the motion) $F = be^{av}$ where b and a are constants and v is the velocity. At $t = 0$ it is moving with velocity v_0, find the velocity at later times.

Solution:

The motion is one dimensional (only in the vertical axis), therefore we take all the quantities to be the scalar components it the direction. The units are $a = \frac{sec}{m}$, $b = \frac{kg \cdot m}{sec^2}$. Using Newton's second law, $F = ma$, and the definition for acceleration $a = \frac{dv}{dt}$:

$$-be^{av} = m\frac{dv}{dt}$$

$$-\frac{b}{m}dt = e^{-av}dv$$

using initial condition $v(t = 0) = v_0$, we integrate

$$\int_0^t -\frac{b}{m}dt = \int_{v_0}^{v} e^{-av}dv$$

$$-\frac{b}{m}t = \frac{-1}{a} \cdot (e^{-va} - e^{-v_0 a})$$

$$e^{-va} = \frac{ab}{m}t + e^{-v_0 a}$$

$$\ln\left(\frac{ab}{m}t + e^{-v_0 a}\right) = -va$$

$$v = -\frac{1}{a}\ln\left(\frac{ab}{m}t + e^{-v_0 a}\right),$$

we find

$$v = \frac{1}{a}\ln\left(\frac{m}{abt + m \cdot e^{-v_0 a}}\right)$$

we should check that the units are correct

$$\frac{\mathrm{L}}{\mathrm{T}} \cdot \ln\left(\frac{\mathrm{M}}{\frac{\mathrm{TML}}{\mathrm{LT}^2} \cdot \mathrm{T} + \mathrm{M} \cdot e^{-\frac{\mathrm{LT}}{\mathrm{TL}}}}\right) = \frac{\mathrm{L}}{\mathrm{T}}$$

we should also preform a sanity check: as $t \to \infty$ we find that the $v \to -\infty$, which is not right for a retarding force, the reason for that is that the force has no direction once the particle reaches $v = 0$, thus the solution is only valid for $v \geq 0$.