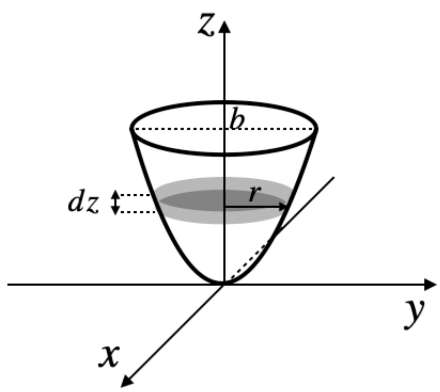
Tutorial 6

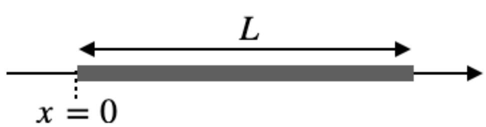
1 Center of Mass

Find the center of mass for the following bodies:

1. A paraboloid $z = a\left(x^2 + y^2\right)$ between $z = 0$ and $z = b$, with a uniform density ρ.



2. A rod with Length L, with density $\lambda = \alpha x^2$, where $x = 0$ is at the edge of the rod.



Solution:

Center of mass (CM) is defined as

$$\boldsymbol{r}_{CM} = \frac{\int \rho\left(\boldsymbol{r}\right) \boldsymbol{r}\, dV}{\int \rho\left(\boldsymbol{r}\right)\, dV},$$

where $\rho\left(r\right)$ is the density of the object and dV is a differential volume element.

1. We write down the expression for the CM, which is now simplified due to the uniform density

$$\boldsymbol{r}_{CM} = \frac{\not{\rho} \int \boldsymbol{r}\, dV}{\not{\rho} \int dV}.$$

We will use cylindrical coordinate system, which is composed of polar coordinates on the $x - y$ plane with the additional z axis: (r, φ, z), so that the paraboloid is expressed as $z = ar^2$. Volume element in this coordinate system is

$$dV = r d\varphi\, dr\, dz.$$

We slice the paraboloid into thin disks at each z, so that each slice has an area of πr^2. It is clear that the CM of each disk is located at its center, due to symmetry, thus $\boldsymbol{r}_{CM}$ is in the direction of $\hat{\boldsymbol{z}}$. Therefore, all we need to do us sum over all the slices using $r^2 = z/a$:

$$\begin{aligned} \boldsymbol{r}_{CM} &= \frac{\int (z\hat{\boldsymbol{z}})\, r d\varphi\, dr\, dz}{\int r d\varphi\, dr\, dz} \\ &= \frac{\pi/a \int_0^b z^2\, dz}{\pi/a \int_0^b z\, dz} \hat{\boldsymbol{z}} \\ &= \frac{b^3/3}{b^2/2} \hat{\boldsymbol{z}} \\ \boldsymbol{r}_{CM} &= \frac{2}{3} b \hat{\boldsymbol{z}}. \end{aligned}$$

2. Since this is a one dimensional integral we simply write

$$\begin{aligned} x_{CM} &= \frac{\int_0^L x\lambda(x)\, dx}{\int_0^L \lambda(x)\, dx} \\ &= \frac{\int_0^L \alpha x^3 dx}{\int_0^L \alpha x^2 dx} \\ &= \frac{L^4/4}{L^3/3} \\ x_{CM} &= \frac{3}{4} L. \end{aligned}$$

2 Men and Flatcar

N men, each with mass m, stand on a railway flatcar of mass M.
They jump off one end of the flatcar with velocity $\boldsymbol{u}$ relative to the car.
The car rolls in the opposite direction without friction.

1. What is the final velocity of the flatcar if all the men jump off at the same time?
2. What is the final velocity of the flatcar if they jump off one at a time? (The answer can be left in the form of a sum of terms)
3. Does case (1) or case (2) yield the larger final velocity of the flatcar? Can you give a simple physical explanation for your answer?

Solution:

There are 2 possible solutions, depending on the definition of $\boldsymbol{u}$:
If u is the velocity relative to the velocity of the flatcar before the jump then:

1. N jump at once

$$\boldsymbol{P}_{initial} = 0$$

$$\boldsymbol{P}_{final} = M\boldsymbol{v}_f + Nm\boldsymbol{u}$$

From momentum conservation

$$\boldsymbol{P}_{initial} = \boldsymbol{P}_{final}$$

and we get

$$\boldsymbol{v}_{f,a} = -\frac{Nm}{M}\boldsymbol{u}$$

2. One jump at a time: Let the speed of the flatcar be $\boldsymbol{v}_j$ after j of N have jumped. Now our system includes only the flatcar and the men who haven't jump yet.

$$\boldsymbol{P}_{initial} = \left[(N-j)\,m + M\right]\boldsymbol{v}_j$$

After 1 more jump

$$\boldsymbol{P}_{final} = \left[(N-j-1)\,m + M\right]\boldsymbol{v}_{j+1} + m\left(\boldsymbol{v}_j + \boldsymbol{u}\right)$$

There are no external forces, so between jumps $\Delta \boldsymbol{P} = 0$

$$\begin{aligned}\Delta \boldsymbol{P} &= \boldsymbol{P}_{j+1} - \boldsymbol{P}_j = \\ &= \left[(N-j-1)\,m + M\right]\boldsymbol{v}_{j+1} + m\left(\boldsymbol{v}_j + \boldsymbol{u}\right) - \left[(N-j)\,m + M\right]\boldsymbol{v}_j \\ &= \left[(N-j-1)\,m + M\right]\boldsymbol{v}_{j+1} + m\boldsymbol{u} - \left[(N-j-1)\,m + M\right]\boldsymbol{v}_j = 0\end{aligned}$$

$$\boldsymbol{v}_{j+1} = \boldsymbol{v}_j - \frac{m}{(N-j-1)\,m + M}\boldsymbol{u}$$

So after N jumps we get

$$\boldsymbol{v}_{f,b} = -\left(\frac{m}{(N-1)\,m + M} + \frac{m}{(N-2)\,m + M} + ... + \frac{m}{M}\right)\boldsymbol{u}$$

3. Comparing the magnitudes of the final velocities we found for both cases:

$$v_{f,a} = \left(\frac{m}{M} + \frac{m}{M} + ... + \frac{m}{M}\right)u > v_{f,b}$$

In the trivial special case N = 1, case (1) and case (2) are identical. Note that case (2) is closely analogous to the derivation of rocket motion.In case (2), however, the expelled mass is in finite packets, one man at a time, while for the rocket the expelled mass is a continuous flow. To help understand why the flatcar moves faster in case (1), think of scenario (2): each man that jumps off the flatcar grants momentum to the flatcar and the remaining men on it, but it also takes away some of the momentum that was given to him by the previous jumpers, as his final velocity is $\boldsymbol{v}_j + \boldsymbol{u}$. Compared to the simultaneous jump for which all N men grant the same momentum to the flatcar alone.

If u is the velocity relative to the velocity of the flatcar right before the jumper disconnects with it then:

In each situation, the flatcar does not accelerate further after the jumper leaves.
Just as the jumper leaves, his speed is the final speed of the flatcar minus the speed relative to the flatcar.
We can always choose a frame of reference in which the flatcar and its load are initially at rest.
We want to find the final velocity of the flatcar after N jumps $\boldsymbol{v}_f$.

1. N jump at once

$$\boldsymbol{P}_{initial} = 0$$

$$\boldsymbol{P}_{final} = M\boldsymbol{v}_f + Nm\left(\boldsymbol{v}_f + \boldsymbol{u}\right)$$

From momentum conservation

$$\boldsymbol{P}_{initial} = \boldsymbol{P}_{final}$$

and we get

$$\boldsymbol{v}_{f,a} = -\left(\frac{Nm}{Nm + M}\right)\boldsymbol{u}$$

2. One jump at a time: Let the speed of the flatcar be $\boldsymbol{v}_j$ after j of N have jumped. Now our system includes only the flatcar and the men who haven't jump yet.

$$\boldsymbol{P}_{initial} = \left[(N-j)\,m + M\right]\boldsymbol{v}_j$$

After 1 more jump

$$\boldsymbol{P}_{final} = [(N-j-1)\,m + M]\,\boldsymbol{v}_{j+1} + m\,(\boldsymbol{v}_{j+1} + \boldsymbol{u})$$

There are no external forces, so between jumps $\Delta \boldsymbol{P} = 0$

$$\begin{aligned}\Delta \boldsymbol{P} = \boldsymbol{P}_{j+1} - \boldsymbol{P}_j &= \\ &= ([(N-j-1)\,m + M]\,\boldsymbol{v}_{j+1} + m\,(\boldsymbol{v}_{j+1} + \boldsymbol{u})) - [(N-j)\,m + M]\,\boldsymbol{v}_j \\ &= [(N-j)\,m + M]\,\boldsymbol{v}_{j+1} + m\boldsymbol{u} - [(N-j)\,m + M]\,\boldsymbol{v}_j = 0\end{aligned}$$

$$\boldsymbol{v}_{j+1} = \boldsymbol{v}_j - \frac{m}{(N-j)\,m + M}\boldsymbol{u}$$

So after N jumps we get

$$\boldsymbol{v}_{f,b} = -\left(\frac{m}{Nm+M} + \frac{m}{(N-1)\,m+M} + ... + \frac{m}{m+M}\right)\boldsymbol{u}$$

3. Comparing the magnitudes of the final velocities we found for both cases:

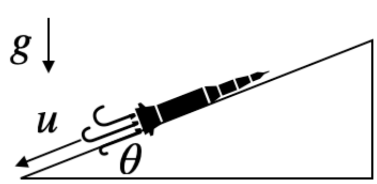
$$v_{f,a} = \left(\frac{m}{Nm+M} + \frac{m}{Nm+M} + ... + \frac{m}{Nm+M}\right)u < v_{f,b}$$

In the trivial special case N = 1, case (1) and case (2) are identical. Note that case (2) is closely analogous to the derivation of rocket motion.In case (2), however, the expelled mass is in finite packets, one man at a time, while for the rocket the expelled mass is a continuous flow. To help understand why the flatcar moves faster in case (2), assume that the mass of the flatcar is small. In this situation, when the men jump together the flatcar moves forward at speed slightly less than u, and the men are moving slowly with respect to the ground. This result is nearly independent of the number of men jumping. Consider now case (2), when the men jump one at a time. The last jumper by himself could cause the forward speed of the flatcar to be close to u, but if there are several jumpers, each previous jumper also contributes to increasing the speed of the flatcar. In case (2), therefore, the final speed of the flatcar could exceed u.

3 Rocket On a Track

A rocket with initial mass m_0 placed at rest on a track with angle θ. At $t = 0$ the rocket begins ejecting mass in the downward direction of the slope. The rate of mass loss is γ, and the velocity of the ejected mass (relative to the rocket) is u, both constants in time. The static and kinetic coefficients of friction, between the rocket and the track, are $\mu_k = \mu_s = \mu$.

The rocket also experiences gravitational acceleration g.



1. What is the condition for the ejection rate of the mass, so that the rocket will immediately begin motion at $t = 0$?

2. Assuming that the rocket begins motion at $t = 0$, what is the acceleration of the rocket?

Solution:

The mass of the rocket is time dependent, and changes as

$$m(t) = m_0 - \gamma t.$$

A short calculation of the change in momentum for an object which ejects mass shows

$$\boldsymbol{p}(t) = m\boldsymbol{v}$$
$$\boldsymbol{p}(t+dt) = \underbrace{(m-dm)(\boldsymbol{v}+d\boldsymbol{v})}_{\text{remaining mass}} + \underbrace{dm\ (\boldsymbol{v}+\boldsymbol{u})}_{\text{ejected mass}},$$

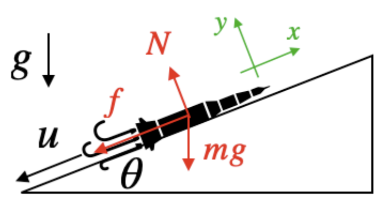
where $\boldsymbol{v}$ is the initial velocity (before ejection), $\boldsymbol{u}$ is the velocity of the ejected mass and $d\boldsymbol{v}$ is the change in the velocity of the remaining mass. Therefore the change in momentum is

$$d\boldsymbol{p} = \boldsymbol{p}(t+dt) - \boldsymbol{p}(t) = m d\boldsymbol{v} + dm\,\boldsymbol{u},$$

where $\boldsymbol{u}_{rel}$ is the velocity of dm relative to m. So that the change in momentum is

$$\frac{d\boldsymbol{p}}{dt} = m\boldsymbol{a} + \left|\frac{dm}{dt}\right|\boldsymbol{u}$$

We choose our coordinate system to coincide with the plane and mark all the forces on the rocket:



Next, we write down the equations of motion for each direction

$$x: \quad -mg\sin\theta - f = \frac{dp_x}{dt} = ma - |\dot{m}|\,u,$$
$$y: \quad N - mg\cos\theta = 0.$$

1. In order for the rocket to begin motion at $t = 0$ we demand that

$$f = \mu N, \quad a = 0$$

   So that the limit occurs at the static configuration

$$-m_0 g(\sin\theta + \mu\cos\theta) = -\gamma_c u \quad \to \quad \gamma_c = \frac{m_0 g}{u}(\sin\theta + \mu\cos\theta),$$

   where we require $\gamma > \gamma_c$ in order to have $a \neq 0$.

2. For an accelerating configuration, in which $\gamma > \gamma_c$, we find

$$-mg(\sin\theta + \mu\cos\theta) = ma - \gamma u,$$

   or

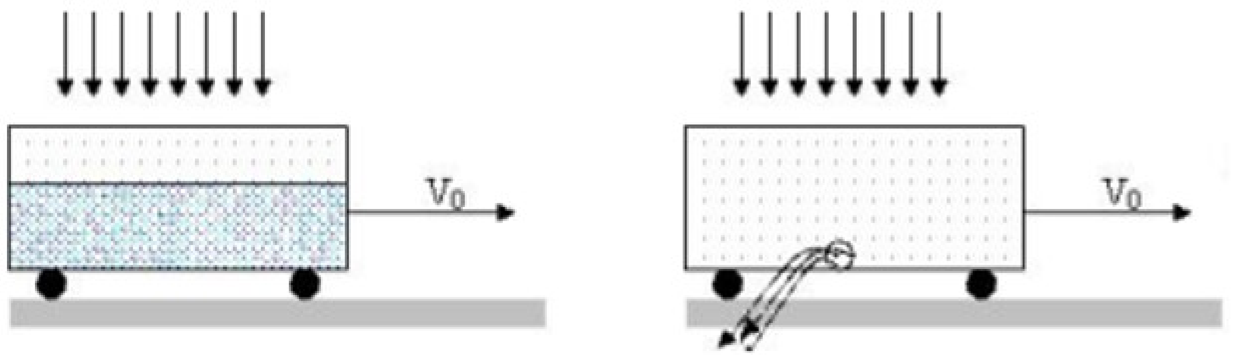
$$a = \frac{u\gamma}{m} - g(\sin\theta + \mu\cos\theta),$$

   where m is time dependent as mentioned at the beginning of the question.

4 Rainy Day

On a rainy day, two identical carriages are sent at an equal initial speed $\overrightarrow{V_0} = V_0\hat{x}$.
In one car the rainwater accumulates at a constant rate, and in the other car there is a hole that allows the rainwater to flow out perpendicular to the direction of movement.
The friction between the carriages and the track is negligible.
The rate of water accumulation / loss is α.

1. Write an expression for the velocity of each carriage as a function of time.
2. Which carriage will travel a greater distance until its final stop?
3. An external force $\overrightarrow{F} = F\hat{x}$ is suddenly applied to the carriage without the hole.
   Calculate the speed of the carriage as a function of time.
   Check the boundaries of which $\alpha \to 0$, $F \to 0$.



Solution:

1. Starting from the carriage without the hole marked with index "1".
   In this solution, we use a coordinate system of which $\hat{y}$ axis directed up and $\hat{z}$ axis pointing toward us.
   Let's write the change in momentum between two random moments during the motion:

$$\overrightarrow{P}(t) = m(t)\overrightarrow{V}(t) + \Delta m\overrightarrow{v}$$

$$\overrightarrow{P}(t + \Delta t) = (m(t) + \Delta m)\left(\overrightarrow{V}(t) + \Delta\overrightarrow{V}(t)\right)$$

   Where, $\overrightarrow{V}(t)$ and $\Delta\overrightarrow{V}(t)$ are the momentary velocity of the carriage and its change in Δt both directed in the $\hat{x}$ direction. and $\overrightarrow{v} = -v\hat{y}$ is the raindrops velocity.
   And $\Delta m = \alpha\Delta t$.

$$\Delta\overrightarrow{P} = m(t)\Delta\overrightarrow{V}(t) + \Delta m\left(\overrightarrow{V}(t) + \Delta\overrightarrow{V}(t) - \overrightarrow{v}\right)$$

   Now divide by Δt and taking it to be infinitesimally small. We get

$$\frac{d\overrightarrow{P}}{dt} = m(t)\frac{d\overrightarrow{V}}{dt} + \frac{dm}{dt}\left(\overrightarrow{V} - \overrightarrow{v}\right)$$

   Since there are no force acting in the $\hat{x}$ direction $\frac{dP_x}{dt} = 0$.

So looking only on the $\hat{x}$ component of the equation and taking $m(t) = m_0 + \alpha t$ we get

$$(m_0 + \alpha t)\frac{dV_x}{dt} = -\alpha V_x$$

$$\frac{dV_x}{V_x} = -\frac{\alpha}{m_0 + \alpha t}dt$$

Taking the integral between $t = 0$ and some later t.

$$\int_{V_0 0}^{V_x(t)} \frac{dV_x}{V_x} = -\int_0^t \frac{\alpha}{m_0 + \alpha t}dt$$

In the right-hand side (RHS) let's take $a = m_0 + \alpha t \Rightarrow da = \alpha dt$

$$\ln\left(\frac{V_x}{V_0}\right) = -\ln\left(\frac{m_0 + \alpha t}{m_0}\right)$$

$$V_x = \frac{m_0}{m_0 + \alpha t}V_0$$

Now for the carriage with the hole. Notice that the mass in it stay constant.

$$\overrightarrow{P}(t) = m\overrightarrow{V}(t) + \Delta m\overrightarrow{v}$$

$$\overrightarrow{P}(t + \Delta t) = m\left(\overrightarrow{V}(t) + \Delta\overrightarrow{V}(t)\right) + \Delta m\overrightarrow{V}(t)$$

In $\overrightarrow{P}(t+\Delta t)$ the first term is the total mass in the carriage at $t+\Delta t$ moving with the new momentary velocity, and the second term is the mass lost due to the hole with the velocity of an earlier moment.

$$\Delta\overrightarrow{P} = m\Delta\overrightarrow{V}(t) + \Delta m\left(\overrightarrow{V}(t) + \Delta\overrightarrow{V}(t) - \overrightarrow{v}\right)$$

In the same manner as for the first carriage:

$$\frac{d\overrightarrow{P}}{dt} = m\frac{d\overrightarrow{V}}{dt} + \frac{d(\alpha t)}{dt}\left(\overrightarrow{V} - \overrightarrow{v}\right)$$

$$m\frac{dV_x}{dt} = -\alpha V_x$$

$$\frac{dV_x}{V_x} = -\frac{\alpha}{m_0}dt$$

$$\ln\left(\frac{V_x}{V_0}\right) = -\frac{\alpha}{m_0}t$$

$$V_x = V_0 e^{-\frac{\alpha}{m_0}t}$$

2. $e^{-\frac{\alpha}{m_0}t}$ goes faster to zero than $\frac{m_0}{m_0+\alpha t}$ therefore the carriage with the hole will stop first.
3. Applying a constant force on the carriage without the hole - the x component of the momentum is changing according Newton's second law $\frac{d\overrightarrow{P}}{dt} = \overrightarrow{F}$

$$\frac{d\overrightarrow{P}}{dt} = m(t)\frac{d\overrightarrow{V}}{dt} + \frac{dm}{dt}\left(\overrightarrow{V} - \overrightarrow{v}\right) = \overrightarrow{F}$$

$$(m_0 + \alpha t)\frac{dV_x}{dt} = F - \alpha V_x$$

$$\frac{dV_x}{F - \alpha V_x} = \frac{dt}{m_0 + \alpha t}$$

$$-\frac{1}{\alpha} \ln \left(\frac{F - \alpha V_x}{F - \alpha V_0} \right) = \frac{1}{\alpha} \ln \left(\frac{m_0 + \alpha t}{m_0} \right)$$

$$\frac{F - \alpha V_x}{F - \alpha V_0} = \frac{m_0}{m_0 + \alpha t}$$

$$V_x = \frac{Ft + m_0 V_0}{m_0 + \alpha t}$$

Taking $\alpha \to 0$

$$V_x \to V_0 + \frac{F}{m_0} t$$

We get the velocity of a body with mass m_0 that affected by a constant force.
Taking $F \to 0$ we get the same answer as in (1):

$$V_x \to \frac{m_0}{m_0 + \alpha t} V_0$$