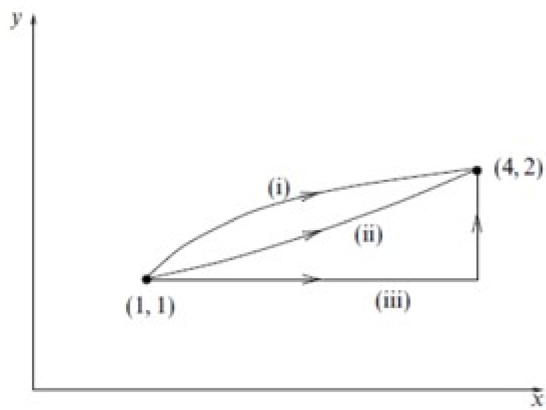
Tutorial 7

1 Work as path integral

Calculate the work by the force $\boldsymbol{F} = (x+y)\,\hat{\boldsymbol{x}} + (y-x)\,\hat{\boldsymbol{y}}$ along the following trajectories:

1. The parabola $y^2 = x$ between $(1,1)$ to $(4,2)$.
2. The curve $x = 2u^2 + u + 1$, $y = 1 + u^2$ between $(1,1)$ to $(4,2)$.
3. The line $y = 1$ between $(1,1)$ to $(4,1)$ and then the line $x = 4$ between $(4,1)$ to $(4,2)$.



Solution:

Our trajectories are in the $x-y$ plane so $\boldsymbol{dl} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}}$ and we want to find a single variable to help us express dx, dy, and $\boldsymbol{F}$.

1. We can choose our variable to be y and if we set the limits to be $y : 1 \to 2$ so $(x,y) = \left(y^2, y\right) : (1,1) \to (4,2) \Rightarrow$

$$W = \int_{(1,1)}^{(4,2)} \boldsymbol{F} \cdot \boldsymbol{dl} = \int_1^2 \left(y + y^2, y - y^2\right) \cdot (2y dy, dy) =$$
$$= \int_1^2 2\left(y^2 + y^3\right) dy + \int_1^2 \left(y - y^2\right) dy = \int_1^2 \left(2y^3 + y^2 + y\right) dy =$$
$$= \left[2\frac{y^4}{4} + \frac{y^3}{3} + \frac{y^2}{2}\right]_1^2 = 11\frac{1}{3}\text{ J}.$$

2. We can choose our variable to be u and the trajectory element vector becomes

$$\begin{matrix} x = 2u^2 + u + 1 \\ y = 1 + u^2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} dx = (4u+1)\,du \\ dy = 2udu \end{matrix}$$

One can also find the limits on u by setting $u = 0$ where $(x, y) = (1, 1)$ and setting $u = 1$ where $(x, y) = (4, 2)$. Then we get $u : 0 \to 1$.

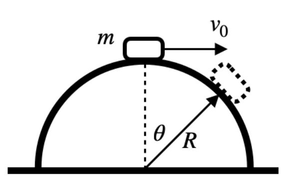
$$W = \int_{(1,1)}^{(4,2)} \boldsymbol{F} \cdot \boldsymbol{dl} = \int_0^1 \left(3u^2 + u + 2, -u - u^2\right) \cdot (4u + 1, 2u)\, du =$$
$$= \int_0^1 \left[\left(3u^2 + u + 2\right)(4u + 1) - \left(u + u^2\right) 2u\right] du = \int_0^1 \left[10u^3 + 5u^2 + 9u + 2\right] du =$$
$$= 10\frac{2}{3}\,\text{J}.$$

3. Here we separate our trajectory into 2 parts. In the first one our variable will be x while $y = const \Rightarrow dy = 0$, and in the second part y is the variable and $x = const \Rightarrow dx = 0$.

$$W = \int_{(1,1)}^{(4,2)} \boldsymbol{F} \cdot \boldsymbol{dl} = \int_1^4 \boldsymbol{F} \cdot \boldsymbol{dx} + \int_1^2 \boldsymbol{F} \cdot \boldsymbol{dy} =$$
$$= \int_1^4 1 + x\, dx + \int_1^2 y - 4\, dy = \left[x + \frac{x^2}{2}\right]_1^4 + \left[\frac{y^2}{2} - 4y\right]_1^2 = 10.5 - 2.5$$
$$= 8\,\text{J}.$$

2 Sliding on Dome

A body with mass m is sliding on a spherical dome with radius R, under the influence of gravity. It is known that the body begins its motion with initial velocity v_0, as shown in the figure.



1. What is the work done by the gravitational force as a function of the angle θ to the body?
2. What is the kinetic energy of the body at angle θ?
3. What are the radial and tangent accelerations?
4. At what angle will the body part from the dome?

Solution:

1. Setting the zero potential energy to be at at the base of the dome, we can write the potential energy as

$$W_g = \int m\boldsymbol{g} \cdot d\boldsymbol{\ell} = \int_0^\theta mgR \sin\left(\theta'\right)\, d\theta' = mgR\left(1 - \cos\theta\right).$$

2. Because the only force, with a component in the direction of the motion of the body is gravity, we find

$$\Delta E_k = \sum_i W_i = mgR\left(1 - \cos\theta\right)$$

thus

$$E_k\left(\theta\right) = \frac{1}{2}mv_0^2 + mgR\left(1 - \cos\theta\right).$$

3. Since the only force tangent to the direction of motion is mg, we find, using $\boldsymbol{F} = m\boldsymbol{a}$, that

$$a_T = g\sin\theta.$$

Whereas the radial acceleration is $a_r = -R\dot{\theta}^2$ (since $\dot{r} = \ddot{r} = 0$), thus

$$a_r = -\frac{v^2}{R} = -\frac{2}{mR}E_k\left(\theta\right) = -\frac{2}{mR}\left[\frac{1}{2}mv_0^2 + mgR\left(1 - \cos\theta\right)\right] = -\left[\frac{v_0^2}{R} + 2g\left(1 - \cos\theta\right)\right].$$

4. The body parts from the dome when the normal force vanishes, i.e. $N = 0$, which yields

$$-mg\cos\theta = ma_r \quad \rightarrow \quad g\cos\theta = \frac{v_0^2}{R} + 2g\left(1 - \cos\theta\right).$$

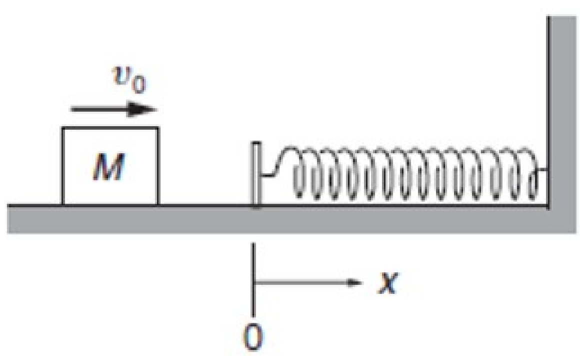
Solving for θ, we find

$$\cos\theta = \frac{1}{3}\left(\frac{v_0^2}{gR} + 2\right).$$

Checking our result we see that when $\frac{v_0^2}{gR} > 1$ there is no solution for θ. This because the body does not slide on the dome right from the start, instead it shoots off into the air immediately.

3 Block, Spring, and Friction

A block of mass M slides along a horizontal table with speed v_0.
At $t = 0$ it hits a spring with spring constant k and begins to experience a friction force, as indicated in the sketch.
The coefficient of friction is variable and is given by $\mu = bx$, where b is a constant.
Find the distance l the block travels before coming to rest.



Solution:

Initial kinetic energy:

$$K_i = \frac{1}{2}Mv_0^2$$

Final kinetic energy is vanishes because we are looking for $x = l$ where the mass velocity is zero.

$$K_f = 0$$

The force on M

$$\overrightarrow{F} = \overrightarrow{F}_{friction} + \overrightarrow{F}_{spring} = -\mu N\hat{x} - kx\hat{x}$$

where $N = Mg$ and we get

$$\overrightarrow{F} = -\left(Mgbx + kx\right)\hat{x}$$

This is a problem in 1D so the Work-Energy Theorem becomes

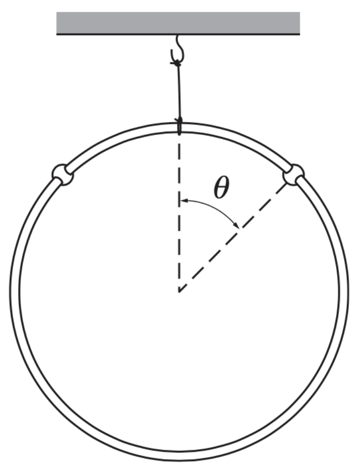
$$\Delta K = \int F dx$$

$$-\frac{1}{2}Mv_0^2 = -\int_0^l (Mgbx + kx)dx = -\left(\frac{Mgb + k}{2}\right)\left[x^2\right]_0^l = -\frac{Mgb + k}{2}l^2$$

$$l = \sqrt{\frac{Mv_0^2}{k + Mgb}}.$$

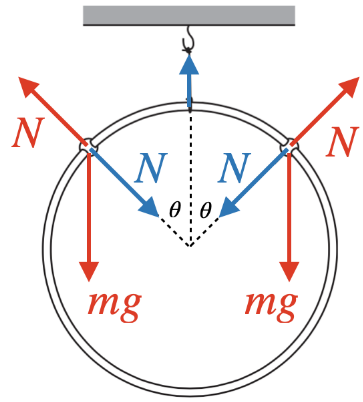
4 Beads on Hanging Ring

A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction, as shown in the figure. The beads are released simultaneously from the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > 3M/2$, and find the angle at which this occurs.



Solution:

Since there is no friction, the mechanical energy is conserved. Let us mark the forces on each of the beads (red) and on the ring (blue)



The condition for the ring to remain stationary is $T \geq 0$ (since if it is not so, the string would be loose). Writing the equations of motion for the ring and the beads yields

$$T - 2N\cos\theta - Mg = 0,$$

$$N - mg\cos\theta = -m\frac{v^2}{R},$$

which comprise 4 unknown parameters: N, T, v and θ, but we are interested in the domain $T \leq 0$. Therefore we need only one additional condition, we turn to change in the kinetic energy:

$$\Delta E_k = \sum_i W_i = mgR\left(1 - \cos\theta\right)$$

$$E_k = \frac{1}{2}mv^2 = mgR\left(1 - \cos\theta\right),$$

hence

$$v^2 = 2gR\left(1 - \cos\theta\right).$$

Plugging this result into the bead's equation, we find

$$N = mg\left(3\cos\theta - 2\right),$$

using $T \leq 0$ along with the ring's equation yields

$$2N\cos\theta + Mg \leq 0 \quad \rightarrow \quad \cos^2\theta - \frac{2}{3}\cos\theta + \frac{1}{6}\frac{M}{m} \leq 0.$$

Solving for $\cos\theta$ we find

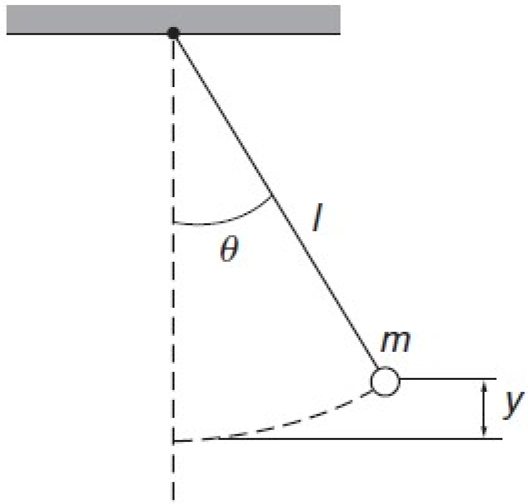
$$\cos\theta = \frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{1}{6}\frac{M}{m}},$$

which has a solution only if $\frac{1}{9} - \frac{1}{6}\frac{M}{m} \geq 0$, i.e. $m \geq \frac{3}{2}M$. In order to find θ_c we consider the solution for $\cos\theta$, we must take the $+$ sign since we are interested in the smallest θ for which the ring rises ($\cos\theta$ decrease with the increase of θ). This

$$\cos\theta = \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{1}{6}\frac{M}{m}}.$$

5 Pendulum Motion

A simple pendulum - a point mass M hanging from a massless string of length l - moves in a circular arc in a vertical plane.
Find $\theta(t)$ using energy considerations given that the maximal angle of the pendulum swing is θ_0 and that at $t = 0$, $\theta = 0$.
*You can assume $\theta_0 \ll 1$.



Solution:

The kinetic energy for θ_0

$$K_{\theta_0} = 0$$

because that's where the motion changes direction.
At some different θ

$$K_\theta = \frac{1}{2}M\left(l\dot{\theta}\right)^2$$

The forces on M are tension $T(-\sin\theta, \cos\theta)$ for some θ of the motion, and the gravitational force $Mg(0,-1)$
The location of the mass is given by $\overrightarrow{r} = l(\sin\theta, 1-\cos\theta)$ then the element of the trajectory of the mass is given by $l(\cos\theta, \sin\theta)\,d\theta$
Using the work-energy theorem

$$\Delta K = \frac{1}{2}M\left(l\dot{\theta}\right)^2 = \int\limits_{\theta_0}^{\theta}\left[T(-\sin\theta, \cos\theta) + (0, -Mg)\right]\cdot(\cos\theta, \sin\theta)\,ld\theta =$$

$$= \int\limits_{\theta_0}^{\theta} -Mg\sin\theta\,ld\theta = Mgl(\cos\theta - \cos\theta_0)$$

For small angles $\cos\theta \approx 1 - \frac{1}{2}\theta^2$
And we get the differential equation

$$\frac{1}{2}M\left(l\dot{\theta}\right)^2 = \frac{1}{2}Mgl\left(\theta_0^2 - \theta^2\right)$$

$$\dot{\theta} = \sqrt{\frac{g}{l}}\sqrt{\theta_0^2 - \theta^2}$$

$$\frac{\frac{d\theta}{\theta_0}}{\sqrt{1 - \left(\frac{\theta}{\theta_0}\right)^2}} = \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = \sqrt{\frac{g}{l}}dt$$

The integral on the left has the form $\int dx/\sqrt{1-x^2} = \arcsin x$, where $x = \frac{\theta}{\theta_0}$

$$\arcsin \frac{\theta}{\theta_0} - 0 = \sqrt{\frac{g}{l}}\,(t-0)$$

$$\theta(t) = \theta_0 \sin\left(\sqrt{\frac{g}{l}}t\right)$$

Notice that $\omega = \sqrt{\frac{g}{l}}$ is the frequency of the pendulum.