Tutorial 8

1 Conservative Force

1. Given the following force
$$\boldsymbol{F} = f_0 \left(y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}} \right),$$
   (a) Is the force conservative?
   (b) A particle positioned at $(R, 0)$ is moving along a circle with radius R counterclockwise, up to the point $(0, R)$. What is the work done by the force?
   (c) Repeat (b) when the particle moves backwards.

2. Given the following force
$$\boldsymbol{F} = -k \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\left(x^2 + y^2 + z^2\right)^{3/2}},$$
   (a) Show that the force is conservative (you may ignore the singularity at $(0, 0, 0)$).
   (b) Find the potential function U for the force.
   (c) A body moves from position $(0, 0, L)$ to the position (L, L, L) in a straight line. Calculate the work done, using path integral.
   (d) Repeat (c), this time using the potential function.

Solution:

1. For the force $\boldsymbol{F} = f_0 \left(y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}} \right)$:
   (a) In order to find if $\boldsymbol{F}$ is conservative we evaluate its rotor,
$$\boldsymbol{\nabla} \times \boldsymbol{F} = \left(\partial_x F_y - \partial_y F_x\right) \hat{\boldsymbol{z}} = \left(-f_0 - f_0\right) \hat{\boldsymbol{z}} = -2f_0\hat{\boldsymbol{z}} \neq 0,$$
   thus $\boldsymbol{F}$ is not a conservative force.
   (b) The work done by $\boldsymbol{F}$ is
$$W = \int \boldsymbol{F} \cdot d\boldsymbol{r}.$$
   Let us evaluate the integral in polar coordinates:
$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \\ \hat{\boldsymbol{x}} &= \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{y}} &= \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}, \end{aligned}$$

which reads

$$\begin{aligned}
\boldsymbol{F} &= f_0\left[R\sin\theta\left(\cancel{\cos\theta\hat{\boldsymbol{r}}} - \sin\theta\hat{\boldsymbol{\theta}}\right) - R\cos\theta\left(\cancel{\sin\theta\hat{\boldsymbol{r}}} + \cos\theta\hat{\boldsymbol{\theta}}\right)\right] \\
&= -f_0R\left[-\sin^2\theta\hat{\boldsymbol{\theta}} - \cos^2\theta\hat{\boldsymbol{\theta}}\right] \\
&= -f_0R\hat{\boldsymbol{\theta}}.
\end{aligned}$$

We also need to write $d\boldsymbol{r}$ properly as

$$\begin{aligned}
d\boldsymbol{r} &= dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} \\
&= (dr\cos\theta - rd\theta\sin\theta)\,\hat{\boldsymbol{x}} + (dr\sin\theta + rd\theta\cos\theta)\,\hat{\boldsymbol{y}} \\
&= dr\,(\cos\theta\hat{\boldsymbol{x}} + \sin\theta\hat{\boldsymbol{y}}) + rd\theta\,(\cos\theta\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{x}}) \\
&= dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}},
\end{aligned}$$

in our case the motion is only in the angular direction, therefore $dr = 0$, i.e. $d\boldsymbol{r} = Rd\theta\,\hat{\boldsymbol{\theta}}$. Therefore,

$$\begin{aligned}
W &= \int \boldsymbol{F}\cdot d\boldsymbol{r} \\
&= -\int_0^{\pi/2}\left(f_0R\hat{\boldsymbol{\theta}}\right)\cdot\left(Rd\theta\,\hat{\boldsymbol{\theta}}\right) \\
&= -\int_0^{\pi} f_0R^2d\theta \\
&= -\frac{\pi}{2}f_0R^2.
\end{aligned}$$

(c) Since the only thing that changes is the direction of the motion (the force remain the same in each point on the path) then the work is simply

$$W = +\frac{\pi}{2}f_0R^2.$$

2. For the force $\boldsymbol{F} = -k\frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{3/2}}$,

(a) In order to find if $\boldsymbol{F}$ is conservative we evaluate its rotor,

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{F} &= (\partial_yF_z - \partial_zF_y)\,\hat{\boldsymbol{x}} + (\partial_zF_x - \partial_xF_z)\,\hat{\boldsymbol{y}} + (\partial_xF_y - \partial_yF_x)\,\hat{\boldsymbol{z}} \\
&= \left(\cancel{-k\frac{z\,(-3/2)\,2y}{(x^2+y^2+z^2)^{5/2}}} + \cancel{k\frac{y\,(-3/2)\,2z}{(x^2+y^2+z^2)^{5/2}}}\right)\hat{\boldsymbol{x}} + (\dots)\,\hat{\boldsymbol{y}} + (\dots)\,\hat{\boldsymbol{z}} \\
&= 0.
\end{aligned}$$

(b) We need to find U such that $\boldsymbol{F} = -\boldsymbol{\nabla}U$. Looking at the x direction

$$F_x = -k\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\partial_xU,$$

integration yields

$$U(x,y,z) = \int k\frac{x}{(x^2+y^2+z^2)^{3/2}}dx$$

$$\begin{cases} s \equiv x^2+y^2+z^2 \\ ds = 2xdx \end{cases}$$

$$= \frac{k}{2}\int \frac{1}{s^{3/2}}ds$$

$$= -k\frac{1}{s^{1/2}} + C_x(y,z)$$

$$U(x,y,z) = -k\frac{1}{(x^2+y^2+z^2)^{1/2}} + C_x(y,z).$$

Following the same routine for y and z yields

$$U(x,y,z) = -k\frac{1}{(x^2+y^2+z^2)^{1/2}} + C_y(x,z) \quad \text{and} \quad U(x,y,z) = -k\frac{1}{(x^2+y^2+z^2)^{1/2}} + C_z(x,y),$$

which means $C_x(y,z) = C_y(x,z) = C_z(x,y)$. The only possible solution is a constant C, thus the potential is

$$U(x,y,z) = -k\frac{1}{(x^2+y^2+z^2)^{1/2}} + C,$$

and since a constant shift in the potential does not affect physical results, which depend only on potential differences, we may write it simply as

$$U(x,y,z) = -k\frac{1}{(x^2+y^2+z^2)^{1/2}}.$$

Another way:
As before, we first use the equation in the x direction

$$F_x(x,y,z) = -\frac{\partial}{\partial x}U(x,y,z) \quad \rightarrow \quad U(x,y,z) = -\int F_x dx = -\frac{k}{(x^2+y^2+z^2)^{1/2}} + h(y,z),$$

where $h(y,z)$ is any general function of y and z, so that be calculating dU/dx we always get back F_x since $dh(y,z)/dx = 0$.
Next we turn to the equation for the y direction

$$F_y(x,y,z) = -\frac{\partial}{\partial y}U(x,y,z) = -\frac{\partial}{\partial y}\left[-\frac{k}{(x^2+y^2+z^2)^{1/2}} + h(y,z)\right] = -k\frac{y}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial y}h(y,z),$$

plugging in the expression for $F_y$we find

$$-k\frac{y}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial y}h(y,z) = -k\frac{y}{(x^2+y^2+z^2)^{3/2}} \quad \rightarrow \quad \frac{\partial}{\partial y}h(y,z) = 0,$$

thus

$$h(y,z) = g(z),$$

where $g(z)$ is a function of z alone, since $h(y,z)$ is a function of y and z alone and $\partial h/\partial y = 0$.
Finally we turn to the equation in the z direction

$$F_z(x,y,z) = -\frac{\partial}{\partial z}U(x,y,z) = -\frac{\partial}{\partial z}\left[-\frac{k}{(x^2+y^2+z^2)^{1/2}} + g(z)\right] = -k\frac{z}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial z}g(z),$$

plugging in the expression for F_z we find

$$-k\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{\partial}{\partial z}g\left(z\right)=-k\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\quad\rightarrow\quad\frac{\partial}{\partial z}g\left(z\right)=0,$$

thus

$$g\left(z\right)=C,$$

where C is a constant. Therefore, after using all three equations we find

$$U\left(x,y,z\right)=-\frac{k}{\left(x^2+y^2+z^2\right)^{1/2}}+C,$$

for which we can choose $C=0$, since a constant does not affect physical quantities which depend on energy differences alone.

(c) Although the required path is a straight line, we've found that the force $\boldsymbol{F}$ is conservative, thus we may evaluate the work in any desired path - the simplest is by moving in two segments $(0,0,L)\rightarrow(0,L,L)\rightarrow(L,L,L)$:

$$\begin{aligned}
W&=\int\boldsymbol{F}\cdot d\boldsymbol{r}\\
&=\int_0^L\boldsymbol{F}\cdot d\boldsymbol{y}+\int_0^L\boldsymbol{F}\cdot d\boldsymbol{x}\\
&=\int_0^L\left(-k\frac{0\cancel{\hat{\boldsymbol{x}}}+y\hat{\boldsymbol{y}}+L\cancel{\hat{\boldsymbol{z}}}}{\left(0^2+y^2+L^2\right)^{3/2}}\right)\cdot d\boldsymbol{y}+\int_0^L\left(-k\frac{x\hat{\boldsymbol{x}}+L\cancel{\hat{\boldsymbol{y}}}+L\cancel{\hat{\boldsymbol{z}}}}{\left(x^2+L^2+L^2\right)^{3/2}}\right)\cdot d\boldsymbol{x}\\
&=-\int_0^L k\frac{y}{\left(y^2+L^2\right)^{3/2}}dy-k\int_0^L\frac{x}{\left(x^2+2L^2\right)^{3/2}}dx\\
&=-k\left[\left.\frac{-1}{\left(y^2+L^2\right)^{1/2}}\right|_0^L+\left.\frac{-1}{\left(x^2+2L^2\right)^{1/2}}\right|_0^L\right]\\
&=-k\left[\cancel{\frac{-1}{\sqrt{2L^2}}}+\frac{1}{L}+\frac{-1}{\sqrt{3L^2}}+\cancel{\frac{1}{\sqrt{2L^2}}}\right]\\
&=\frac{k}{L}\left(\frac{1}{\sqrt{3}}-1\right).
\end{aligned}$$

(d) Using the potential, all we need to do is evaluate the potential difference

$$W=U\left(0,0,L\right)-U\left(L,L,L\right)=-k\frac{1}{\left(L^2\right)^{1/2}}+k\frac{1}{\left(3L^2\right)^{1/2}}=\frac{k}{L}\left(\frac{1}{\sqrt{3}}-1\right),$$

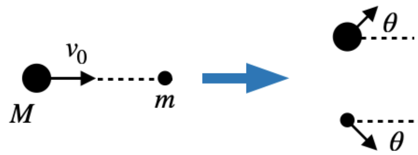
which is the same as (c).

2 Scattering

A body with mass $M=\alpha m$ moves with velocity v_0 towards a stationary body with mass m. After the collision between the two bodies, each body moves at opposite direction with the same angle relative to the original direction of motion (see figure).

Notice: the velocities are not necessarily equal.

Given that the collision between the bodies was elastic (energy was conserved), and that the process happens on a horizontal plane,



1. Assume that $\alpha = 2$. What is θ?
2. What is the maximum possible ration $\alpha = M/m$ for such scattering?

Solution:

1. Since all the forces act on the bodies are internal (within the system of the two bodies), the total momentum is conserved. Therefore

$$\boldsymbol{p}_i = (\alpha m v_0, 0)$$
$$\boldsymbol{p}_f = (P_x + p_x, P_y + p_y)\,,$$

thus

$$\alpha m v_0 = P_x + p_x \quad P_y = -p_y.$$

Since the directions are opposite, we have

$$\left|\frac{P_y}{P_x}\right| = \left|\frac{p_y}{p_x}\right| \quad \rightarrow \quad P_x = p_x,$$

$$P_x = p_x = \frac{1}{2}\alpha m v_0.$$

In order to find the y direction quantities we use energy conservation,

$$\frac{1}{2}\alpha m v_0^2 = \frac{P_x^2 + P_y^2}{2\alpha m} + \frac{p_x^2 + p_y^2}{2m}$$

$$P_y^2 = \frac{\alpha^2 m^2 v_0^2}{4}\left(\frac{3-\alpha}{1+\alpha}\right) \quad \rightarrow \quad P_y = \pm\frac{\alpha m v_0}{2}\sqrt{\frac{3-\alpha}{1+\alpha}} \quad \rightarrow \quad \tan\theta = \frac{P_y}{P_x} = \pm\sqrt{\frac{3-\alpha}{1+\alpha}}.$$

Then, for $\alpha = 2$,

$$\tan\theta = \pm\sqrt{\frac{1}{3}} \quad \rightarrow \quad \theta = \pm\frac{\pi}{6}.$$

2. In order to find the maximum possible ration we require $\alpha < 3$, thus

$$0 < \frac{M}{m} < 3.$$

3 Pendulum's Potential

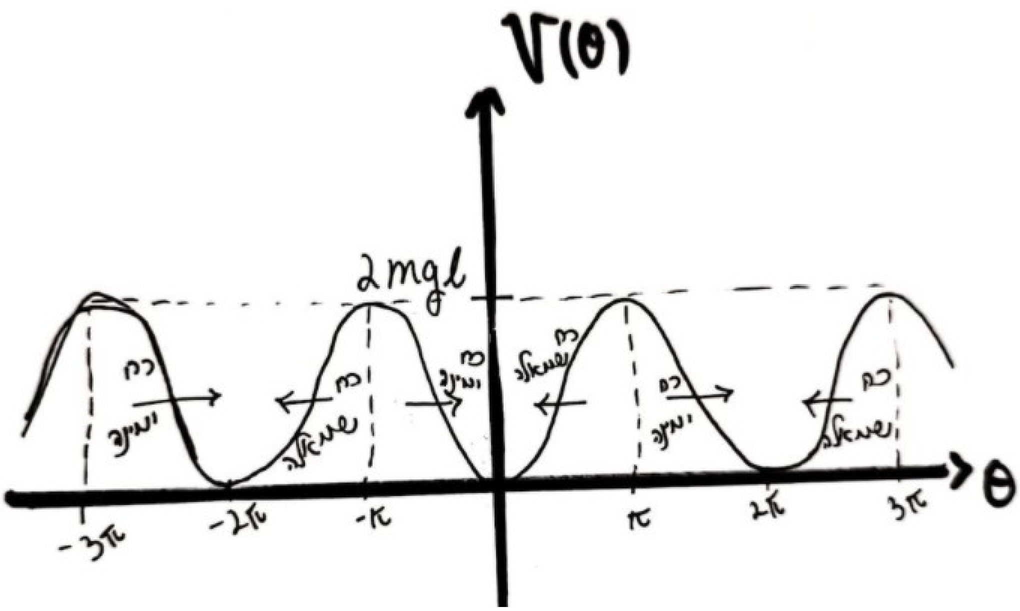
A pendulum of a thin and mass-less rod of length l ,which attached at its upper end to a static point and at its lower end to a mass m, is free to rotate around the static point.

1. What force are acting on the mass? Are they conservative?
2. What is the potential energy V of the mass, as a function of the angle of the pendulum θ?
3. Sketch $V(\theta)$.

4. Mark on the graph you have made the regions; where the force of the mass is towards $\hat{\theta}$, and where the force is towards $-\hat{\theta}$.
5. Are there equilibrium points? Are they stable?
6. What kind of trajectories the mass could have? Are they bounded (between two turning points)?
7. The kinetic energy of the mass at the bottom of the pendulum is $K(\theta = 0) = K_0$.
   If there are bounded trajectories, What is the limits on the value of K_0 for a bounded trajectory?

Solution:

1. The forces on the pendulum are the gravitational force mg - a conservative force, and the tension of the rod T.
   T is not necessarily a conservative force but since he doesn't do work on the mass it is meaningless for our goals.
2. V is the gravitational potential energy $V(\theta) = mgh(\theta) = mgl\,(1 - \cos\theta)$.
3. The potential can be sketched as



4. The force is related to the potential energy by
$$\overrightarrow{F} = -\frac{dV(\theta)}{ld\theta}\hat{\theta}$$
   so the force is towards $\hat{\theta}$ when the slope is negative and the force is towards $-\hat{\theta}$ when the slope is positve.
5. When $\frac{dV(\theta)}{ld\theta} = 0$ the total force on the mass is zero - equlibrium.
   Let's find the equilibrium points where $-\pi \leq \theta \leq \pi$ (the physical region):
$$0 = \frac{dV(\theta)}{ld\theta} = mg\sin\theta$$
   and we get the points $\theta = 0, \pi, -\pi$.
   $\theta = 0$ is a stable equlibrium - after a little displacement, a force will bring you back to the same point.
   While $\theta = \pi \iff \theta = -\pi$ is unstable - after a little displacement, a force will drive the mass further away.

6. The body can perform two types of trajectories. For $E > 2mgl$ the kinetic energy is larger the the potential energy for any $\theta$and the body will preform a circular motion.
For $E < 2mgl$ the body will oscillate around the stable point $\theta = 0$ - bounded.

7. The maximal energy for bounded trajectory is $K_0 = 2mgl$. As long as $K_0 < 2mgl$ the body will move in a closed path.

4 Lennard-Jones Potential

A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones potential given by

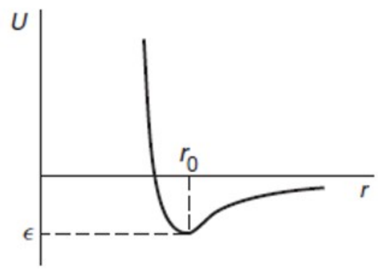
$$U = \epsilon \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right]$$

1. Find the position of the potential minimum and its value.

2. Near the minimum the atoms execute simple harmonic motion. Find the frequency of oscillation.

3. Find a range of values for the system energy for which atoms will move in bounded trajectories.

4. For a given energy E that is in the range you found above, find the distances r between the two atoms that are turning points.



Solution:

1. Differentiating U

$$\frac{dU}{dr} = \epsilon \left[-12 \left(\frac{r_0^{12}}{r^{13}} \right) + 12 \left(\frac{r_0^6}{r^7} \right) \right]$$

$\frac{dU}{dr} = 0$ for $r = r_0$.
Substituting $r = r_0$ in U

$$U(r_0) = -\epsilon.$$

2. In order to find the frequency of oscillations we will use the spring model near the minimum point r_0, where

$$U(r) = \underbrace{U(r_0)}_{\text{constant}} + \cancel{U'(r = r_0)}(r - r_0) + \frac{1}{2}U''(r = r_0)(r - r_0)^2 \cdots \approx \frac{1}{2}k\Delta r^2 \quad \rightarrow \quad k = U''(r = r_0)$$

and since we know that, for the spring model,

$$\omega^2 = \frac{k}{\mu},$$

where μ is the mass of oscillating object, thus

$$\omega = \sqrt{\left(\frac{d^2U}{dr^2}\right)_{r=r_0} \frac{1}{\mu}}.$$

Since the potential depends on the distance between the two atoms it is easier to choose the frame of reference of one of these atoms, so that the distance between the atoms $\boldsymbol{r} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ will be our new coordinate and we can deal only with the motion of one atom instead of both. Doing so we must take into account that the relative acceleration $\boldsymbol{a}_{\text{rel}}$ as well. Let us note that the second and third Newtons laws read

$$\boldsymbol{F}_{12} = m_1 \ddot{\boldsymbol{r}}_1 \quad \boldsymbol{F}_{21} = m_1 \ddot{\boldsymbol{r}}_2 \quad \text{and} \quad \boldsymbol{F}_{12} = -\boldsymbol{F}_{21},$$

hence

$$\ddot{\boldsymbol{r}}_2 = -\frac{m_1}{m_2}\ddot{\boldsymbol{r}}_1.$$

Therefore, the relative acceleration is

$$\boldsymbol{a}_{\text{rel}} = \ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = \left(1 + \frac{m_1}{m_2}\right)\ddot{\boldsymbol{r}}_1 = \frac{m_1 + m_2}{m_1 m_2}\underbrace{m_1 \ddot{\boldsymbol{r}}_1}_{\boldsymbol{F}_{12}} = \frac{m_1 + m_2}{m_1 m_2}\boldsymbol{F}_{12}.$$

Thus, the motion of atom m_1 is described by the equation

$$\boldsymbol{F}_{12} = \frac{m_1 m_2}{m_1 + m_2}\boldsymbol{a}_{\text{rel}} \quad \Longleftrightarrow \quad \boldsymbol{F} = \mu \boldsymbol{a},$$

where $\mu = \frac{m_a}{2}$ and $m_a = m_1 = m_2$ is the mass of each atom.
Getting back to the expression for ω,

$$\left(\frac{d^2U}{dr^2}\right)_{r=r_0} = \epsilon\left[156\left(\frac{r_0^{12}}{r^{14}}\right) - 84\left(\frac{r_0^6}{r^8}\right)\right]_{r=r_0} = \epsilon\,[156 - 84] = 72\epsilon$$

And we get

$$\omega = \sqrt{\frac{144\epsilon}{m_a}}.$$

3. For $r \to \infty$ $U \to 0$. Therefore, for energies in the range $-\epsilon < E < 0$ the motion of the atoms will be bounded.

4. We need to find the distances between the two atoms where $U(r) = E\,(< 0)$. $E = -|E|$ and $|E| < \epsilon$

$$-|E| = \epsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right]$$

Multiplying by r^{12}

$$-|E|\,r^{12} + 2\epsilon r_0^6 r^6 - \epsilon r_0^{12} = 0$$
$$r^{12} - 2\frac{\epsilon}{|E|}r_0^6 r^6 + \frac{\epsilon}{|E|}r_0^{12} = 0$$

This is a quadratic equation for r^6.

$$r^6 = \left[\frac{\epsilon}{|E|}r_0^6 \pm \sqrt{\left(\frac{\epsilon}{|E|}\right)^2 r_0^{12} - \frac{\epsilon}{|E|}r_0^{12}}\right] = \frac{\epsilon}{E}r_0^6\left[1 \pm \sqrt{1 - \frac{|E|}{\epsilon}}\right].$$