# Tutorial 12

## 1 Flight Attendant

A plane with a rest length $L_0 = 100m$ flies at a speed of $0.8c$.
Inside the plane there is a flight attendant moving at a speed $0.55c$ relative to the plane, from the tail of the plane towards its nose.

1. What is the speed of the flight attendant relative to earth?
2. According to her clock, how long does it takes for the flight attendant to cross the plane?
3. How long will it take for the flight attendant to cross the plane according to an observer on Earth?
4. At 10 o'clock the flight attendant left the tail of the plane with a snack cart and $0.3\,\mu sec$ later (according to the flight attendant's clock) a passenger located $34m$ from the nose of the plane (according to the pilot) opened a bag of peanuts.
Is it possible that the flight attendant gave these peanuts to the passenger?

**Solution:**

1. Let's define two frames: $S$ - earth and $S'$ - the plane.
Denote $v_p = 0.8c$ for the speed of the plane relative to earth, $u' = 0.55c$ for the speed of the flight attendant relative to the plane, and $u$ for the speed of the flight attendant relative to earth.
Lorentz factor between $S$ and $S'$ is given by
$$\Gamma = \frac{1}{\sqrt{1-0.8^2}} = \frac{5}{3}$$
Use the reverse Lorentz transformation
$$u = \frac{dx}{dt} = \frac{\not{\Gamma}\left(dx' + \frac{v_p}{c}cdt'\right)}{\not{\Gamma}\left(dt' + \frac{v_p}{c^2}dx'\right)} = \frac{\not{dt'}\left(\frac{dx'}{dt'} + v_p\right)}{\not{dt'}\left(1 + \frac{v_p}{c^2}\frac{dx'}{dt'}\right)} = \frac{u' + v_p}{1 + \frac{v_p u'}{c^2}}$$
So
$$u = 0.9375c$$
2. Lorentz factor between the plane and the flight attendant is given by
$$\gamma = \frac{1}{\sqrt{1-0.55^2}} \cong 1.2.$$
**Solution 1:** According to length contraction the length of the plane according to the attendant
$$L' = \frac{L_0}{\gamma} \simeq 83.3m$$
therefore it will takes $t'' = \frac{L_0}{\gamma u'} = 0.506\,\mu sec$ for the flight attendant to cross the plane.
**Solution 2:** For an observer standing in the plane the time it takes for the flight attendant to travel from the tail to the nose is given by $t' = \frac{L_0}{u'} = 0.606\,\mu sec.$
According to time dilation in the attendant frame this time is contracted (and by a Lorentz factor) and is given by $\gamma t'' = t' \Rightarrow t'' = \frac{L_0}{u'\gamma} = 0.506\,\mu sec.$

3. Knowing the time in the flight attendant frame (rest frame for her crossing the plane) we can use time dilation to find the time duration as an observer on earth measure it $t$, by:

$$t = \frac{1}{\sqrt{1-0.9375^2}} t'' = 1.45\,\mu sec.$$

4. Distance between the passenger and the tail of the plane in the plane's frame $\Delta x = 100m - 34m = 66m$.
   In the attendant's frame using length contraction

$$\Delta x' = \frac{\Delta x}{\gamma} = 55.12m$$

   But the length traveled by the flight attendant is $u' \cdot 0.3\,\mu sec = 49.5m$.
   We conclude it isn't possible that the passenger got his peanuts from the flight attendant.

## 2 Photon Collision

A Photon with an energy of $E_0$ collides elastically with a particle with a mass $m_0$ that is at rest.
If the photon scatters with an angle $\theta$ relative to its original direction of motion, What will be the scattering angle of the particle?

**Solution:**

Recall:
Energy of a massive particle is given by $E = \sqrt{(pc)^2 + (m_0c^2)^2}$, while for non-massive particle (such as photon) $E = pc$.
In an elastic collision, energy and momentum is conserved.
Denote $\varphi$ for the scattering angle of the particle.
x - component of momentum conservation:

$$\frac{E_0}{c} = p_{photon}\cos\theta + p_{particle}\cos\varphi$$

y - component of momentum conservation

$$0 = p_{photon}\sin\theta - p_{particle}\sin\varphi$$

Taking $\frac{p_y}{p_x}$

$$\tan\varphi = \frac{p_{photon}\sin\theta}{\frac{E_0}{c} - p_{photon}\cos\theta}$$

Isolate $p_{particle}$ in the momentum equations

$$p_{particle}\cos\varphi = \frac{E_0}{c} - p_{photon}\cos\theta$$

$$p_{particle}\sin\varphi = p_{photon}\sin\theta$$

Taking $p_x^2 + p_y^2$

$$p_{particle}^2 = \frac{E_0^2}{c^2} - 2\frac{E_0 p_{photon}}{c}\cos\theta + p_{photon}^2$$

Energy conservation

$$E_0 + m_0c^2 = p_{photon}c + \sqrt{(p_{particle}c)^2 + (m_0c^2)^2}$$

$$E_0 + m_0c^2 - p_{photon}c = \sqrt{(p_{particle}c)^2 + (m_0c^2)^2}$$

Taking a power of 2

$$E_0^2 + \cancel{m_0^2 c^4} + p_{photon}^2 c^2 + 2\left(E_0 m_0 c^2 - E_0 p_{photon} c - m_0 p_{photon} c^3\right) = p_{particle}^2 c^2 + \cancel{m_0^2 c^4}$$

$$p_{particle}^2 = \frac{1}{c^2}\left[E_0^2 + p_{photon}^2 c^2 + 2\left(E_0 m_0 c^2 - E_0 p_{photon} c - m_0 p_{photon} c^3\right)\right]$$

Comparing expressions for $p_{particle}^2$

$$\cancel{\frac{E_0^2}{c^2}} - 2\frac{E_0 p_{photon}}{c}\cos\theta + \cancel{p_{photon}^2} = \frac{1}{c^2}\left[\cancel{E_0^2} + \cancel{p_{photon}^2 c^2} + 2\left(E_0 m_0 c^2 - E_0 p_{photon} c - m_0 p_{photon} c^3\right)\right]$$

$$-\frac{\cancel{2}}{c} E_0 p_{photon}\cos\theta = \cancel{2}\left[E_0 m_0 - \frac{E_0}{c} p_{photon} - m_0 p_{photon} c\right]$$

$$m_0 c p_{photon} + \frac{E_0}{c}\left(1 - \cos\theta\right) p_{photon} = E_0 m_0$$

$$p_{photon} = \frac{E_0 m_0 c}{E_0\left(1 - \cos\theta\right) + m_0 c^2}$$

And we can get $\tan\varphi$ by:

$$\tan\varphi = \frac{\frac{E_0 m_0 c}{E_0(1-\cos\theta) + m_0 c^2}\sin\theta}{\frac{E_0}{c} - \frac{E_0 m_0 c}{E_0(1-\cos\theta)+m_0 c^2}\cos\theta} = \frac{\sin\theta}{1-\cos\theta}\frac{m_0 c^2}{E_0 + m_0 c^2}$$

## 3 Levitation by laser light

A 1-kW light beam from a laser is used to levitate a solid aluminum sphere by focusing it on the sphere from below. What is the diameter of the sphere, assuming that it floats freely in the light beam? The density of aluminum is 2.7 g/cm$^3$. *Note*: Power is the defined to be the energy rate $dE/dt$ and is measured in $Watt\,(= W) \equiv J/s$.

**Solution:**

In order to achieve levitation we must require $mg \leq F$, where the force $F$ is the force exerted by the light beam. The force is the change of momentum of the photons in the light beam, which, assuming that the beam is reflected in the same angle (i.e. the radius of the sphere is much larger than the radius of the light beam), is reflected in the same angle $\Delta p = 2p$. Thus, recalling that photon's energy $\epsilon = pc$, we find

$$F = \frac{dp}{dt} = 2\frac{p}{dt} = 2\frac{\epsilon}{cdt} = 2\frac{P}{c} = 2\frac{10^3\,[\mathrm{J/s}]}{3\times 10^8\,[\mathrm{m/s}]} = 6.7\times 10^{-6}\mathrm{N},$$

where $P$ is the power of the light beam. Therefore, taking $\rho = 2.7\times 10^3$ kg/m$^3$, we require

$$F \geq \rho\frac{4\pi}{3} r^3 g \quad \rightarrow \quad r = \left(\frac{3}{4\pi}\frac{F}{\rho g}\right)^{1/3} = 3.9\times 10^{-4}\mathrm{m}.$$

## 4 Doppler shift of a hydrogen spectral line

*Doppler Effect*:
Consider sound waves from a source moving with velocity $v$ through the medium toward an observer at rest. Picture sound as regular series of pulses separated by time $\tau_0 = 1/\nu_0$, where $\nu_0$ is the rate of pulses (1/sec). The distance between the pulses (the wavelength) is $\lambda_0 = v\tau_0 = v/\nu_0$. Usually sound waves are described by a sine wave, for which $\lambda_0$ is the distance between successive peaks and $\nu_0$ is the frequency of

the oscillations. If the source of the sound wave is moving towards the observer at speed $u$, then the distance between successive peaks is $\lambda = \lambda_0 - u\tau_0 = \lambda_0 - u/\nu_0$,

$$\frac{v}{\nu} = \frac{v}{\nu_0} - \frac{u}{\nu_0} \quad \rightarrow \quad \nu = \frac{\nu_0}{1 - u/v},$$

where $\nu$ is the observed frequency. The shift in the frequency (or wavelength) $\Delta\nu = \nu - \nu_0$ is known as the *Doppler shift.*

*Relativistic Doppler effect:*
Considering light signal (light wave propagates with speed $c$) from a moving source at speed $u$. Due to special relativity we know that the time in the observer's frame dilutes, $\tau = \gamma\tau_0$, therefore the observed wavelength is $\lambda = \lambda_0 - u\tau = (c - u)\,\tau$. The observed frequency is

$$\nu = \frac{c}{(c-u)\,\tau} = \frac{1}{1 - u/c}\frac{1}{\gamma\tau_0} = \frac{\nu_0}{1 - u/c}\sqrt{1 - \frac{u^2}{c^2}} = \nu_0\sqrt{\frac{1 + u/c}{1 - u/c}}.$$

One of the most prominent spectral lines of hydrogen is the $H_\alpha$ line, a bright red line with a wavelength of 656.1 nm.

1. What is the expected wavelength of the $H_\alpha$ line from a star receding with a speed of 3000 km/s?
2. The $H_\alpha$ line measured on Earth from opposite ends of the Sun's equator differ in wavelength by $9 \times 10^{-3}$ nm. Assuming that the effect is caused by rotation of the Sun, find the period of rotation. The diameter of the Sun is $1.4 \times 10^6$ km.

**Solution:**

1. The speed of the star is

   $$v = 3 \times 10^6 \frac{\text{m}}{\text{s}},$$

   which is $0.01c$, i.e. 1% of the speed of light. For brevity, let us evaluate the redshift twice: first, without considering relativity effects, then, considering relativity.
   *Without Relativity*:

   $$\lambda = \frac{c}{\nu} = \frac{c}{\nu_0}\left(1 + \frac{v}{c}\right) = \lambda_0\left(1 + \frac{v}{c}\right) = 656.1 \times 10^9\,(1 + 0.01)\,\text{nm} = 662.66\text{nm}.$$

   *With Relativity*:

   $$\lambda = \frac{c}{\nu} = \frac{c}{\nu_0}\sqrt{\frac{1 + v/c}{1 - v/c}} = \lambda_0\sqrt{\frac{1 + v/c}{1 - v/c}} = 656.1 \times 10^9\sqrt{\frac{1 + 0.01}{1 - 0.01}}\text{nm} = 662.69\text{nm}.$$

   We find that the correction due to relativistic effects is of order $10^{-2}$nm, and we may say with confidence that the shift is

   $$\Delta\lambda = \lambda - \lambda_0 = 6.6\text{nm}.$$

2. Let us denote the wavelengths of the two signals by $\lambda_A$ and $\lambda_B$, $\lambda_A < \lambda_B$. The difference in the wavelength, due to Doppler effect is

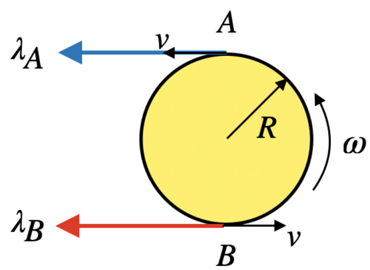
   $$\Delta\lambda = \lambda_B - \lambda_A = \lambda_0\left(\sqrt{\frac{1 + v/c}{1 - v/c}} - \sqrt{\frac{1 - v/c}{1 + v/c}}\right),$$

   using the Taylor expansions

   $$\sqrt{1 \pm x} = 1 \pm \frac{1}{2}x + \dots \qquad \text{and} \qquad \frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{1}{2}x + \dots$$

taking $x = v/c \ll 1$, we may include only the leading order terms

$$\begin{aligned}
\Delta\lambda &= \lambda_0 \left( \sqrt{\frac{1+v/c}{1-v/c}} - \sqrt{\frac{1-v/c}{1+v/c}} \right) \\
&\approx \lambda_0 \left[ \left(1 + \frac{1}{2}\frac{v}{c}\right)\left(1 + \frac{1}{2}\frac{v}{c}\right) - \left(1 - \frac{1}{2}\frac{v}{c}\right)\left(1 - \frac{1}{2}\frac{v}{c}\right) \right] \\
&= \lambda_0 \left[ \cancel{1} + \frac{v}{c} + \cancel{\left(\frac{1}{2}\frac{v}{c}\right)^2} - \cancel{1} + \frac{v}{c} - \cancel{\left(\frac{1}{2}\frac{v}{c}\right)^2} \right] \\
&= 2\lambda_0 \frac{v}{c}
\end{aligned}$$




Therefore

$$v = \frac{\Delta\lambda}{2\lambda_0} c = 6.9 \times 10^{-6} c = 2057.6 \frac{\mathrm{m}}{\mathrm{s}},$$

which corresponds to

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{v} R = \frac{2\pi}{2057.6 \frac{\mathrm{m}}{\mathrm{s}}} \left( \frac{1.4}{2} \times 10^9 \mathrm{m} \right) = 2.14 \times 10^6 \mathrm{s} \approx 24.7 \text{ days}.$$

The sun's period time is actually about 27 days.