Tutorial 13

1 Driven Oscillator

Mass m is in equilibrium with a spring connected to point P and exerts force

$$F = -k(x - l),$$

where x is the length of the spring and l is the rest length of the spring.
At $t = 0$ point P begins to oscillate sinusoidal at amplitude A and frequency ω. Solve the equations of motion for $x(t)$.

Solution:

The displacement of the point P could be written as

$$\Delta(t) = A\sin(\omega t)$$

(with $\Delta(t = 0) = 0$)
In the frame moving with P Newton's second law can be written as

$$\Sigma F = m\ddot{x} = mg - k(x - l) - m\frac{d^2}{dt^2}(A\sin\omega t)$$

where the last term is the fictitious force due to non-inertial frame of reference.
Changing variable according to

$$y = x - l - \frac{mg}{k}$$

and

$$\omega_0^2 = \frac{k}{m}$$

we get an equation of motion of the form of driven harmonic oscillation:

$$\ddot{y} + \omega_0^2 y = \omega^2 A\sin\omega t$$

This is a second-order non-homogeneous differential equation.

Its solution is the sum of the homogeneous solution y_h, which solve

$$\ddot{y_h} + \omega_0^2 y_h = 0,$$

and the private solution y_p, which solve

$$\ddot{y_p} + \omega_0^2 y_p = \omega^2 A \sin \omega t.$$

Starting from the private solution, we will guess solution of the form $y_p(t) = B \sin \omega t$ and set this guess in the equation:

$$-\omega^2 B \sin \omega t + \omega_0^2 B \sin \omega t = \omega^2 A \sin \omega t$$

We will use a coefficient comparison

$$B = \frac{\omega^2}{\omega_0^2 - \omega^2} A$$

The homogeneous solution is simply given by

$$y_h(t) = C \cos \omega_0 t + D \sin \omega_0 t$$

Hence the solution is given by

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = \frac{\omega^2}{\omega_0^2 - \omega^2} A \sin \omega t + C \cos \omega_0 t + D \sin \omega_0 t$$

Finding the constants using initial conditions.
At $t = 0$ there was an equilibrium

$$mg - k(x - l) = 0 \Rightarrow y = 0$$

and

$$\dot{x} = 0 \Rightarrow \dot{y} = 0$$

So using the initial conditions

$$y(t = 0) = 0 \Rightarrow C = 0$$

$$\dot{y}(t = 0) = 0 \Rightarrow \omega_0 D + \frac{\omega^3}{\omega_0^2 - \omega^2} A = 0$$

$$D = -\frac{\omega}{\omega_0} \frac{\omega^2}{\omega_0^2 - \omega^2} A$$

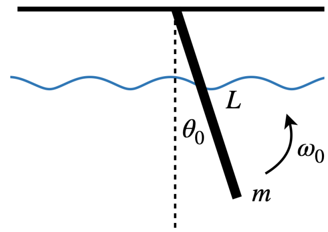
which means that our solution is

$$x(t) = y(t) + l + \frac{mg}{k} =$$

$$= \frac{\omega^2}{\omega_0^2 - \omega^2} A \left(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t \right) + l + \frac{mg}{k}.$$

2 Overdamped Oscillator

A uniform rod with mass m and length L is dipped in a liquid and allowed to rotate around its connection point to the ceiling (see figure). The liquid exerts torque on the rod $\tau = -\beta\omega = -\beta\dot{\theta}$. The rod begins motion from tilting angle θ_i with initial velocity ω_i counterclockwise. Given that $\beta > \sqrt{2ImgL}$, find $\theta(t)$.



Solution:

The torque equation relative to rotation around the connection point to the ceiling reads

$$-mg\frac{L}{2}\sin\theta - \beta\dot{\theta} = I\ddot{\theta} \quad \rightarrow \quad -mg\frac{L}{2}\theta - \beta\dot{\theta} \approx I\ddot{\theta},$$

where we used small angles approximation. Therefore, the equation of motion is

$$\ddot{\theta} + \frac{\beta}{I}\dot{\theta} + \frac{mgL}{2I}\theta = 0.$$

Let us solve for a general equation of this type

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

which can be generally solved by

$$x = Ae^{i\Omega t},$$
$$\dot{x} = i\Omega x,$$
$$\ddot{x} = -\Omega^2 x.$$

Plugging $x(t)$ and its derivatives into the differential equation yields

$$\left(-\Omega^2 + 2i\gamma\Omega + \omega_0^2\right)x = 0 \quad \rightarrow \quad \Omega^2 - 2i\gamma\Omega - \omega_0^2 = 0 \quad \rightarrow \quad \Omega = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2},$$

we find that there is a critical value $\omega_0^2 = \gamma^2$, where $\omega_0^2 > \gamma^2$ yields damped oscillations (underdamping)

$$x(t) = Ae^{-\gamma t}e^{\pm i\omega t} \quad \rightarrow \quad x(t) = Ae^{-\gamma t}\cos(\omega t + \varphi),$$

where $\omega \equiv \sqrt{|\omega_0^2 - \gamma^2|}$. Whereas and $\omega_0^2 < \gamma^2$ yields a completely imaginary Ω, which corresponds to overdamping (no oscillations)

$$x(t) = Ae^{-(\gamma-\omega)t} + Be^{-(\gamma+\omega)t}.$$

In our case

$$2\gamma = \frac{\beta}{I} \qquad \text{and} \qquad \omega_0^2 = \frac{mgL}{2I},$$

thus

$$\beta > \sqrt{2ImgL} = 2I\omega_0 \quad \rightarrow \quad \gamma > \omega_0,$$

there are no oscillations. Recalling that $I = mL^2/3$ gives us

$$\theta(t) = e^{-\gamma t}\left(Ae^{\omega t} + Be^{-\omega t}\right),$$

where $\omega \equiv \sqrt{|\omega_0^2 - \gamma^2|}$ and

$$\gamma = \frac{3}{2}\frac{\beta}{mL^2} \qquad \text{and} \qquad \omega_0^2 = \frac{3}{2}\frac{g}{L}.$$

Plugging in the initial conditions

$$\theta(t=0)=\theta_i=A+B,$$
$$\dot{\theta}(t=0)=\omega_i=-\gamma(A+B)+\omega(A-B),$$

gives us

$$A=\frac{\omega_i+\theta_i(\gamma+\omega)}{2\omega},$$
$$B=-\frac{\omega_i+\theta_i(\gamma-\omega)}{2\omega},$$

thus the solution is

$$\theta(t)=\frac{e^{-\gamma t}}{2\omega}\left[(\omega_i+\theta_i(\gamma+\omega))e^{\omega t}-(\omega_i+\theta_i(\gamma-\omega))e^{-\omega t}\right].$$

3 Driven Damped Oscillator

A mass m is attached to one end of a spring with constant k, which is connected to a vertical wall. The wall moves according to the function $A\sin\Omega t$. The system is submerged in a liquid with viscosity coefficient such that it exerts a force of $-\beta v$ on the mass. It is given that at $t=0$ the spring is relaxed and the mass is stationary.

1. Find the equation of motion for the mass.
2. Assuming underdamped oscillations, express the position of the mass at $t\gg m/\beta$.
3. For what frequency Ω will the system achieve maximal amplitude?

Solution:

1. Let us define x such that the origin is at the relaxation point of the spring at $t=0$, so that the force the spring exerts on the mass is
$$f_e=-k(x-A\sin\Omega t),$$
and the equation of motion is
$$m\ddot{x}=-k(x-A\sin\Omega t)-b\dot{x}\qquad\rightarrow\qquad\ddot{x}+\frac{\beta}{m}\dot{x}+\frac{k}{m}x=\frac{kA}{m}\sin\Omega t.$$

2. Let us define $\gamma\equiv\beta/2m$, $\omega_0^2=k/m$ and $F_0\equiv kA$, so that the equation of motion is
$$\ddot{x}+2\gamma\dot{x}+\omega_0^2x=\frac{F_0}{m}\sin\Omega t.$$
The solution to such equation is composed of the homogeneous and private solutions, where the former is the solution to the equation without the driving force and the latter is a solution which counters the driving force term,
$$x(t)=x_{\mathrm{h}}(t)+x_{\mathrm{p}}(t).$$
The homogeneous solution is that of a damped harmonic oscillator,
$$x_{\mathrm{h}}(t)=\begin{cases}Ae^{-\gamma t}\cos(\omega t+\varphi) & \omega_0^2>\gamma^2\\ e^{-\gamma t}\left(Ae^{\omega t}+Be^{-\omega t}\right) & \omega_0^2<\gamma^2\end{cases}\qquad\text{where}\qquad\omega=\sqrt{|\omega_0^2-\gamma^2|},$$
where in our case we have $\omega_0^2>\gamma^2$. The private solution is found by considering the equation in the complex plane $z=y+ix$, where y is a dummy variably which is only here to help us with the

formalism, taking an equation for y to be $\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = \frac{F_0}{m}\cos\Omega t$, we may write the equation for the complex variable z as the sum of the equations (x equation multiplied by i),

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\Omega t},$$

where at the end we will be only interested in the x component of the solution, that is the imaginary part.

$$\begin{aligned} z_{\mathrm{p}}(t) &= Ce^{i\Omega t}, \\ \dot{z}_{\mathrm{p}}(t) &= iC\Omega e^{i\Omega t}, \\ \ddot{z}_{\mathrm{p}}(t) &= -C\Omega^2 e^{i\Omega t}, \end{aligned}$$

thus

$$-C\Omega^2 e^{i\Omega t} + 2\gamma i C\Omega e^{i\Omega t} + \omega_0^2 C e^{i\Omega t} = \frac{F_0}{m} e^{i\Omega t} \quad \to \quad C = \frac{F_0}{m\left(\omega_0^2 - \Omega^2 + 2i\gamma\Omega\right)},$$

getting rid of the imaginary part in the denominator yields,

$$C = \frac{F_0}{m\left(\omega_0^2 - \Omega^2 + 2i\gamma\Omega\right)} \frac{\omega_0^2 - \Omega^2 - 2i\gamma\Omega}{\omega_0^2 - \Omega^2 - 2i\gamma\Omega} = \frac{F_0\left(\omega_0^2 - \Omega^2 - 2i\gamma\Omega\right)}{m\left(\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2\right)}.$$

In order to write this expression more neatly, we will use the polar form $C = Re^{i\phi}$, where

$$R = \sqrt{CC^*} = \frac{F_0\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}}{m\left(\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2\right)} = \frac{F_0}{m}\sqrt{\frac{1}{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}},$$

and

$$\phi = \arctan\left(\frac{\mathrm{Im}\,[C]}{\mathrm{Re}\,[C]}\right) = -\arctan\left(\frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}\right).$$

Thus the imaginary part of $z = Re^{i(\Omega t + \phi)}$ is our solution for x

$$x_{\mathrm{p}}(t) = \mathrm{Im}\,[z(t)] = R\sin(\Omega t + \phi) = \frac{F_0 \sin(\Omega t + \phi)}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}}, \quad \phi = -\arctan\left(\frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}\right).$$

So the total solution is

$$x(t) = Ae^{-\gamma t}\cos(\omega t + \varphi) + \frac{F_0 \sin(\Omega t + \phi)}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}},$$

where A and φ are found from the initial conditions. However, for $t \gg m/\beta$ the exponent in the left term vanishes, leaving us only with the private solution, which does not depend on the initial conditions.

3. We must maximize the expression for the amplitude, but since Ω appears only in the denominator, let us minimize the denominator instead

$$-4\Omega\left(\omega_0^2 - \Omega^2\right) + 2\,(2\gamma)^2\,\Omega = 0 \quad \to \Omega^2 = \omega_0^2 - 2\gamma^2.$$