HW 1

1 Trigonometric Identities

Euler's formula provides an interpretation of the sine and cosine functions as weighted sums of the exponential function:

$$\cos(x) = Re(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin(x) = Im(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$$

Use these expressions to derive the following identities:

1. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$.
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

Solution:

1.

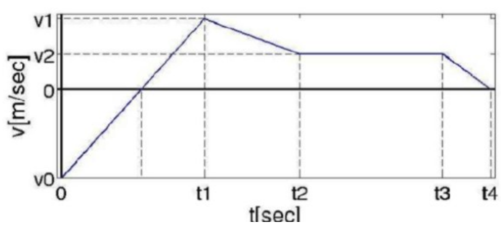
$$\begin{aligned}
\sin(x \pm y) &= \frac{e^{i(x\pm y)} - e^{-i(x\pm y)}}{2i} = \frac{e^{ix}e^{iy} - e^{-ix}e^{\mp iy}}{2i} \\
&\text{(using Euler's formula)} \\
&= \frac{(\cos x + i \sin x)(\cos y \pm i \sin y) - (\cos x - i \sin x)(\cos y \mp i \sin y)}{2i} \\
&= \frac{[\cos x \cos y \mp \sin x \sin y + i(\sin x \cos y \pm \cos x \sin y)]}{2i} \\
&\quad - \frac{[\cos x \cos y \mp \sin x \sin y - i(\sin x \cos y \pm \cos x \sin y)]}{2i} \\
&= \frac{2i(\sin x \cos y \pm \cos x \sin y)}{2i} = \sin x \cos y \pm \cos x \sin y
\end{aligned}$$

2.

$$
\begin{aligned}
\cos(x \pm y) &= \frac{e^{i(x\pm y)} + e^{-i(x\pm y)}}{2} = \frac{e^{ix}e^{iy} + e^{-ix}e^{\mp iy}}{2} \\
&\text{(using Euler's formula)} \\
&= \frac{(\cos x + i\sin x)(\cos y \pm i \sin y) + (\cos x - i \sin x)(\cos y \mp i \sin y)}{2} \\
&= \frac{[\cos x \cos y \mp \sin x \sin y + i(\sin x \cos y \pm \cos x \sin y)]}{2} \\
&\quad + \frac{[\cos x \cos y \mp \sin x \sin y - i(\sin x \cos y \pm \cos x \sin y)]}{2} \\
&= \frac{2(\cos x \cos y \mp \sin x \sin y)}{2} = \cos x \cos y \mp \sin x \sin y
\end{aligned}
$$

2 Trains - High School Mechanics

A train travels in a straight line, it's velocity is given in the graph below (a positive sign indicates movement from south to north):



At $t = 0$ the train passes through a train station.

1. $v_0 = -5\frac{m}{sec}$
2. $v_1 = 4\frac{m}{sec}$
3. $v_2 = 2\frac{m}{sec}$
4. $t_1 = 3\,sec$
5. $t_2 = 5\,sec$
6. $t_3 = 8\,sec$

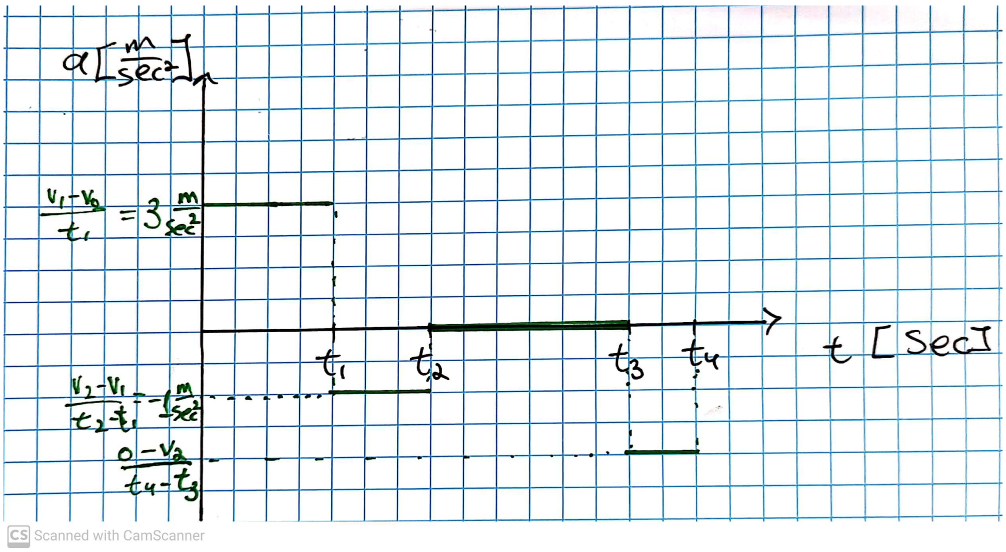
a. Sketch a graph of train acceleration as a function of time.
b. What is the direction of motion at each section?
c. When is the train at the southernmost point of the motion?
What is the distance from the station at this point?

A second train is leaving the same station at time t_1 and travels with a constant velocity of $v_B = 2\frac{m}{sec}$.
d. Calculate t_4 if both trains meet at that time.

e. Where both trains meet relative to the train station?
f. Schematically sketch the first train distance from the station as a function of time.

Solution:

a. The acceleration graph:



b.The train travels south (north) when the velocity is negative (positive).
We need to find when $v(t) = 0$?
Looking at the first section, the slope is given by $\frac{4-(-5)}{3} = 3\,\frac{m}{sec^2}$.

$$v(t) = 3t - 5, \text{ for } 0 \leq t \leq t_1$$

$$v(t = \frac{5}{3}\,sec) = 0$$

So for $0 \leq t \leq \frac{5}{3}$ the train travels south, and for $\frac{5}{3} \leq t \leq t_4$ the train travels north.
c. At $t = \frac{5}{3}\,sec$ the train is at the southernmost point of its motion.
If we set the train station to be in $x = 0$, The location of this point is obtained by setting $t = \frac{5}{3}\,sec$ in the equation of motion:

$$x(t) = -5t + \frac{3}{2}t^2$$

We get that this point is located $\frac{25}{6}\,meter$ away (south direction) from the station.
d. The location of the second train is given by

$$x_B(t) = \underbrace{0}_{\text{location when leaving the station}} + v_B(t - t_1) = 2t - 6$$

for $3\,sec \leq t$.
Now we need to find an expression to the first train's location at $t_3 \leq t$.
In order to find it, we will follow the train's location along the different sections.

We've already found that $x(t = 3\,sec) = -1.5\,meter$. For $t_1 \le t \le t_2$:

$$x(t) = -1.5 + v_1(t - t_1) + \frac{1}{2}\frac{v_2 - v_1}{t_2 - t_1}(t - t_1)^2 =$$
$$= -\frac{1}{2}t^2 + 7t - 18$$

and $x(t = t_2) = 4.5\,meter$. For $t_2 \le t \le t_3$:

$$x(t) = 4.5 + v_2(t - t_2) =$$
$$= -5.5 + 2t$$

so $x(t = t_3) = 10.5\,meter$. Now we can find $x_A(t)$ to express the location of the first train at $t_3 \le t$:

$$x_A(t) = 10.5 + v_2(t - t_3) + \frac{1}{2}\frac{0 - v_2}{t_4 - t_3}(t - t_3)^2 =$$
$$= -5.5 + 2t - \frac{1}{t_4 - 8}(t - 8)^2$$

Both trains meet at t_4 so we can find it by solving $x_A(t_4) = x_B(t_4)$:

$$-5.5 + 2t_4 - \frac{1}{t_4 - 8}(t_4 - 8)^2 = 2t_4 - 6$$

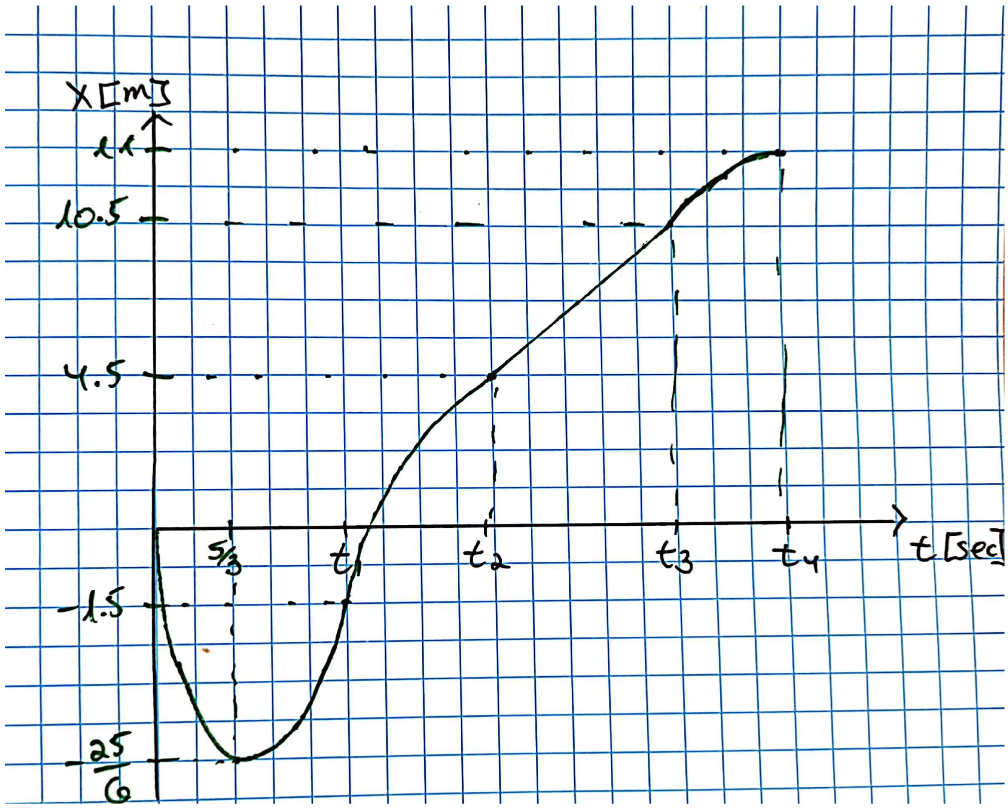
$$0.5 = t_4 - 8$$

$$t_4 = 8.5\,sec$$

e. In order to find where both trains meet we can plug the result we've just found for t_4 into $x_A(t)$ or $x_B(t)$:

$$x_B(t_4) = x_A(t_4) = 2\,8.5 - 6 = 11$$

The trains meet $11\,meters$ north to the station.
f.



3 Wacky Particle

A wacky particle moves along the x axis according to $x = 6\,[\mathrm{m}] + 2\,[\mathrm{m/s}]\,t + 6\,[\mathrm{m/s^2}]\,t^2$.

1. Find the acceleration of the particle.
2. What is the initial velocity and the position of the particle (at $t = 0$)?
3. Find a general expression for the velocity of the particle as a function of time t.
4. What is the velocity of the particle at $t = 4\,[\mathrm{sec}]$?
5. Other particle, just as wacky, accelerates along the x axis according to

$$a = 3\,[\mathrm{m/s^3}]\,t + 2\,[\mathrm{m/s^2}]\,.$$

   Find the expression for the displacement of the particle $x(t)$, while $v(t = 0) = 1\,\mathrm{m/s}$ and $x(t = 0) = 5\,\mathrm{m}$.

Solution:

1. The acceleration of the particle can be found by taking the time derivative of the position twice d^2x/dt^2,

$$a = \frac{d^2x}{dt^2} = \frac{d}{dt}\left[2\,[\mathrm{m/s}] + 12\left[\mathrm{m/s^2}\right]t\right] = 12\,\mathrm{m/s^2},$$

but since the expression for x is a quadratic polynomial of time, hence the particle follows a constant acceleration motion, for which the acceleration is twice the coefficient of the quadratic term (as can also be seen by the units of each coefficient), we could immediately found that $a = 12\,\mathrm{m/s^2}$.

2. Again, we can take the time derivative of x and then plug in $t = 0$, or we can be wiser and immediately pick the coefficient of the linear term. In either case we find $v_i = 2\,\mathrm{m/s}$.

3. In order to find a general expression for the velocity, i.e. a time dependent function $v(t)$ we must take the time derivative of x,

$$v = \frac{dx}{dt} = 2\,[\mathrm{m/s}] + 12\left[\mathrm{m/s^2}\right]t.$$

4. We plug $t = 4\,\mathrm{s}$ into the expression for $v(t)$ to find

$$v(t = 4\,\mathrm{s}) = 2\,[\mathrm{m/s}] + 12\left[\mathrm{m/s^2}\right] \times 4\,[\mathrm{s}] = 50\,\mathrm{m/s}.$$

5. We need to integrate $a(t)$ twice to find $x(t)$. First we find the expression for the velocity

$$v(t) = \int a(t)\,dt = \frac{3}{2}\left[\mathrm{m/s^3}\right]t^2 + 2\left[\mathrm{m/s^2}\right]t + v_i,$$

where the constant v_i is found from the initial condition (i.c.)for the velocity by plugging $t = 0$

$$v(t = 0) = 1\,\mathrm{m/s} = v_i.$$

Next we find the expression for the displacement

$$x(t) = \int v(t)\,dt = \frac{1}{2}\left[\mathrm{m/s^3}\right]t^3 + \left[\mathrm{m/s^2}\right]t^2 + 1\,[\mathrm{m/s}]\,t + x_i,$$

where, again, x_i is found from the i.c.

$$x(t = 0) = 5\,\mathrm{m} = x_i.$$

Therefore

$$x(t) = \frac{1}{2}\left[\mathrm{m/s^3}\right]t^3 + \left[\mathrm{m/s^2}\right]t^2 + 1\,[\mathrm{m/s}]\,t + 5\,\mathrm{m}.$$

4 Elevator

An elevator begin an upward motion with constant acceleration $a = 1$ m/s^2. 1 second after the beginning of the motion, a screw is released and falls down. The height of the elevator is $h = 2.75$m. Find:

1. The time it takes the screw to reach the elevator's floor.
2. The total length of the path, s, the screw undergoes with respect to a static reference frame (someone static outside the elevator).

Solution:

The motion is one dimensional (only in the vertical axis), therefore we take all the quantities to be the scalar components in that direction.

1. Let us break the motion into 2 stages: before and after the release of the screw.

   (a) Before the release of the screw, it moves with the elevator with constant acceleration. Let us find the velocity and displacement it reaches by the time it is released

   $$v(t) = \int a\,dt = at + \cancelto{0}{v_i} = at \quad \rightarrow \quad v(t = 1\text{s}) \equiv u = 1\,\text{m/s},$$
   $$y(t) = \int v\,dt = \frac{a}{2}t^2 + y_i \quad \rightarrow \quad \Delta y(t = 1\text{s}) = y - y_i = 0.5\text{m}.$$

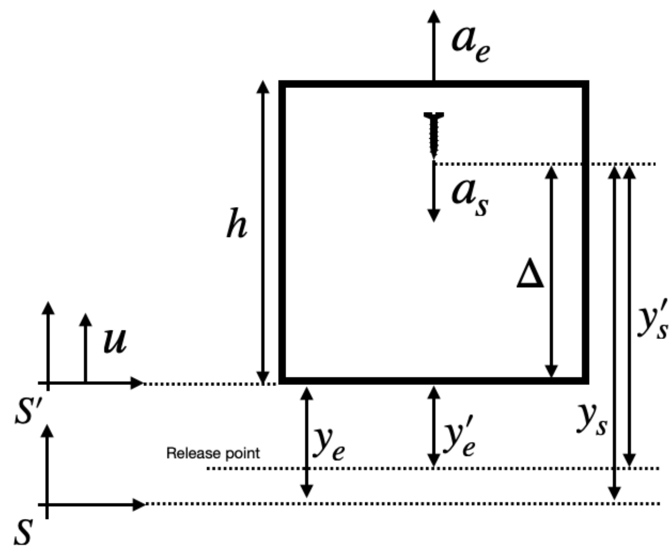
   (b) After the release we treat the elevator and the screw separately. Let us define a new frame of reference S' which moves with constant velocity u and origin's is at the position of the elevator's floor at the time of release, we also set the time to zero at the moment of release, so that

   $$y' = y - \Delta y + ut.$$

   Therefore, the initial positions of the elevator's floor and the screw in the new reference frame are

   $$y'_e(t = 0) = 0,$$
   $$y'_s(t = 0) = h.$$

   This way we don't have to worry about the initial constants from the first stage of the motion (the additional $\Delta y + u$) which are the same for both bodies.
   We look for the time when the distance between the elevator's floor and the screw, $\Delta \equiv y'_s - y'_e$, turns zero (see figure). Thus, all we need to do is find the expressions for y'_e and y'_s and then solve $\Delta(t_f) = 0$ for the falling time t_f.



Both the elevator and the screw move with constant accelerations, thus let us write the expressions for such general motion and then plug in the acceleration for each body. We already know the position expression for motion with constant acceleration

$$y(t) = y_i + v_i t + \frac{a}{2}t^2,$$

plugging in y_i and v_i for each body we find

$$y_e'(t) = \frac{a}{2}t^2,$$
$$y_s'(t) = h - \frac{g}{2}t^2.$$

Therefore

$$\Delta = h - \frac{g}{2}t^2 - \frac{a}{2}t^2,$$

note that if we did not switch reference frames we would have additional constant and linear term in both y_e and y_s which would cancel each other when calculating Δ.
Solving for $\Delta = 0$ we find

$$t = \sqrt{2\frac{h}{a+g}},$$

which is the same time it takes a body to fall a height h with acceleration of $a + g$. Checking units we find $[t] = \left(\mathrm{m} \times \mathrm{s}^2/\mathrm{m}\right)^{1/2} = \mathrm{s}$ which is just dandy. Therefore, the time it takes for the screw to reach the elevator's floor is $t \approx 0.713\mathrm{s}$.

2. In order to find the s we need to calculate the screw's path during the second stage of the motion and add it to the displacement we've found for the first stage. In general, the expression for s is

$$s = \int |v|\, dt = \int_{\text{up}} v dt + \int_{\text{down}} (-v),$$

thus we need to find when the velocity of the screw is positive and when it is negative (with respect to the static reference frame).
Note that the there are two scenarios for the motion of the screw: in one it reaches zero velocity and falls down, while in the other it doesn't have enough time to reach zero velocity as the elevator's floor hit it before that. So we need to find which happens first: the velocity of the screw reaches zero or it hits the floor.
First, we must remember that the initial velocity and position of the screw in a static reference frame are $v_i = 1$ m/s and $x_i = 1$m. We already know the time it takes the screw to hit the floor, while the time it takes it to reach zero velocity is easily calculated as

$$v_s(t) = v_s(t=0) - gt \quad \rightarrow \quad t_0 \equiv t(v=0) = v_i/g \approx 0.1\text{s},$$

which is smaller than the value of the time it takes it to hit the floor. Therefore we need to calculate the screw's displacement during the movement upward, Δ_u, and downward, Δ_d and add the absolute value of them up. The expression for the screw's displacement between time t_1 and t_2 is

$$\Delta_{1\to 2} = y_s(t_2) - y_s(t_1) = v_s(t_1)(t_2 - t_1) - \frac{g}{2}(t_2 - t_1)^2,$$

thus,

$$\Delta_u = u \times 0.1\,[s] - \frac{g}{2}(0.1\,[s])^2 \approx 0.05\text{m},$$
$$\Delta_d = -\frac{g}{2}(0.713\,[s] - 0.1\,[s])^2 \approx -1.84\text{m},$$

and the path during the second stage is

$$|\Delta_u| + |\Delta_d| = 1.89\text{m}.$$

Adding this result to the path from the first stage, which was 0.5m, we find the total path of the screw: 2.39m.

5 Particle moves in 3 dimensions

Particle's position is described by $\boldsymbol{r} = A\,\hat{\boldsymbol{i}} + Bt^2\,\hat{\boldsymbol{j}} + Ct\,\hat{\boldsymbol{k}}$, where $A = 1$m; $B = 4\text{m/s}^2$; $C = 1$m/s.

1. Write the expressions for the velocity of the particle.
2. Write the expressions for the acceleration of the particle.
3. What is the shape of the particle's trajectory?

Solution:

In order to derive the expressions for velocity and acceleration we need to take the derivatives with respect to time:

1. The velocity is $d\boldsymbol{v}/dt$

$$\boldsymbol{v}\left(t\right) = 2Bt\,\hat{\boldsymbol{j}} + C\,\hat{\boldsymbol{k}}.$$

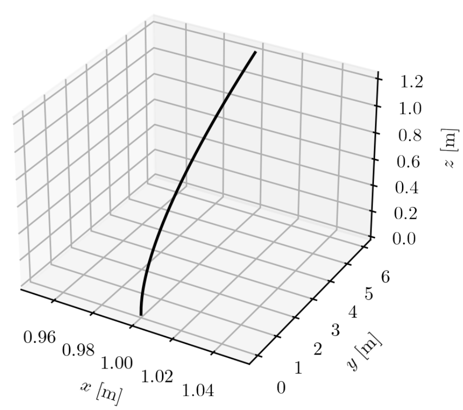
2. The acceleration is $d\boldsymbol{v}/dt$

$$\boldsymbol{a}\left(t\right) = 2B\,\hat{\boldsymbol{j}}.$$

3. In order to describe the shape of the trajectory we look at each direction separately. The $\hat{\boldsymbol{\imath}}$ component of the position vector is constant. Therefore we can look only at the plane which cuts through $A\hat{\boldsymbol{j}}$. The $\hat{\boldsymbol{j}}$ component of the position vector is quadratic with time while the $\hat{\boldsymbol{k}}$ component of the position vector is linear with time. Plugging the expression for t from the $\hat{\boldsymbol{k}}$ component (lets call it z) into the $\hat{\boldsymbol{j}}$ component (lets call it y) expression we find

$$t = \frac{z}{C} \quad \rightarrow \quad y = \frac{B}{C^2}z^2,$$

thus y is parabolic in terms of z:



6 Basketball Physics

In this problem we will explore some of the physics if the 3pt shot in basketball. We'll neglect all air-ball interaction.

Consider a standard NBA basketball court in which the 3pt arc is $D = 7.24$m from the basket, the height of the basket is $H = 3.05$m and a player that release the ball form $h = 2.5$m.

1. Estimate (no calculations!) the best angle in which the player should throw the ball. Remember that we're trying to put a 24cm diameter ball into a 45cm rim. Think about the way the ball "sees" the rim.

2. One of the most important considerations when making any shot is energy consumption. The most energy-efficient shot will be the one thrown at minimal velocity. Why?

3. Calculate $\boldsymbol{v_0}(\theta, h, H, D)$, initial velocity dependence on angle, release height, distance to the basket and rim height.

4. Minimize it with respect to the angle (first do it analytically and only then put in the numbers).
   Which angle is the best now?

5. Read up on Magnus effect. Explain how it could change your conclusions.

Solution:

1. The best angle from this point of view would be the closest to $\pi/2$. This would produce a steep trajectory and allow the ball to utilize the most of the surface of the rim.

2. The chemical energy in the muscles is transferred into kinetic energy. The lower the velocity - the lower the energy.

3. In projectile motion:

$$x(t) = v_0 \cos\theta t$$
$$y(t) = h + v_0 \sin\theta t - \frac{1}{2}gt^2$$

   Combining the two we get:

$$y(x) = h + \tan\theta x - \frac{g}{2v_0^2\cos^2\theta}x^2$$

   Demanding y(x=D)=H we get:

$$v_0 = \sqrt{\frac{g}{2\left(D\tan\theta + h - H\right)}}\frac{D}{\cos\theta}$$

4. In order to minimize v_0 with respect to the angle θ we first need to differentiate the above expression:

$$\frac{dv_0}{d\theta} = \sqrt{\frac{gD^2}{2}}\frac{d}{d\theta}\left(\frac{1}{\cos\theta\sqrt{D\tan\theta + h - H}}\right) =$$
$$= \sqrt{\frac{gD^2}{2}}\frac{-\left(-\sin\theta\sqrt{D\tan\theta + h - H} + \cos\theta\frac{D}{2\cos^2\theta}\frac{1}{\sqrt{D\tan\theta+h-H}}\right)}{\cos^2\theta\left(D\tan\theta + h - H\right)} =$$
$$= \sqrt{\frac{gD^2}{2}}\frac{\sin\theta\left(D\tan\theta + h - H\right) - \frac{D}{2\cos\theta}}{\cos^2\theta\left(D\tan\theta + h - H\right)^{\frac{3}{2}}} = \sqrt{\frac{gD^2}{2}}\frac{D\sin^2\theta - (H-h)\sin\theta\cos\theta - \frac{D}{2}}{\cos^2\theta\left(D\tan\theta + h - H\right)^{\frac{3}{2}}} \Rightarrow$$

Using the trigonometric identities $\sin^2\theta = \frac{1+\cos 2\theta}{2}$ and $2\sin\theta\cos\theta = \sin 2\theta$ we get:

$$\Rightarrow \sqrt{\frac{gD^2}{2}} \frac{D\frac{1+\cos 2\theta}{2} - \frac{1}{2}(H-h)\sin 2\theta - \frac{D}{2}}{\cos^2\theta\,(D\tan\theta + h - H)^{\frac{3}{2}}}$$

This expression is equal zero when:

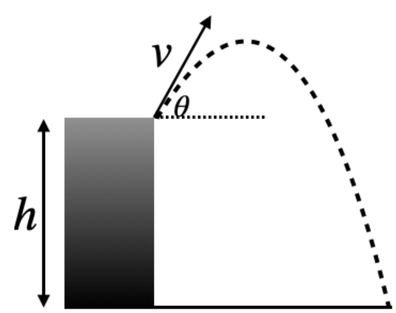
$$D\cos 2\theta = (H-h)\sin 2\theta$$

and we get

$$\theta_{MAX} = \frac{1}{2}\arctan\left(\frac{D}{H-h}\right)$$

7 Projectile motion - Optional

A coin is thrown from a building top at height h, with initial velocity v_0 and angle θ (with respect to the horizon). Show that the final velocity of the coin is independent of θ.



Solution:

Let us calculate the final velocity vector,

$$\boldsymbol{v}_f = \int \boldsymbol{a}dt = -gt\,\hat{\boldsymbol{j}} + \boldsymbol{v}_i,$$

where

$$\boldsymbol{v}_i = v_0\left(\cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}}\right).$$

The norm of $\boldsymbol{v}$ squared is simply

$$|\boldsymbol{v}_f|^2 = (v_0\cos\theta)^2 + (v_0\sin\theta - gt_f)^2,$$

thus we need to express t_f, which is determined by the vertical motion alone:

$$y(t) = y_i + v_i t + \frac{a}{2}t^2 \quad\rightarrow\quad 0 = h + v_0\sin\theta\,t_f - \frac{g}{2}t_f^2 \quad\rightarrow\quad t_f = \frac{v_0\sin\theta}{g} \pm \sqrt{\left(\frac{v_0\sin\theta}{g}\right)^2 + 2\frac{h}{g}},$$

where the "$-$" sign yields a negative value for t_f which is irrelevant to our problem (the time when the coin passed through $y = 0$ before it was at the top of the building). Plugging t_f (with the "+" sign) into the expression for $|\boldsymbol{v}_f|^2$ we find

$$
\begin{aligned}
|\boldsymbol{v}_f|^2 &= (v_0 \cos\theta)^2 + \left(\cancel{v_0 \sin\theta} - g \cancel{\frac{v_0 \sin\theta}{g}} - g\sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + 2\frac{h}{g}} \right)^2 \\
&= \underbrace{(v_0 \cos\theta)^2 + (v_0 \sin\theta)^2}_{v_0^2} + 2gh,
\end{aligned}
$$

thus the norm of the final velocity

$$|\boldsymbol{v}_f| = \sqrt{v_0^2 + 2gh},$$

is independent of the angle θ.