HW 3

1 Vectors and Time Derivatives

The position vector of a particle is given as $\boldsymbol{r} = r\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}} = \cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}$ and $|\boldsymbol{r}| = \text{const}$, calculate:
a. $\frac{d\boldsymbol{r}}{dt}$,
b. $\frac{d}{dt}\left(\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt}\right)$.

What are the meanings of the resulting expressions?

Solution:

a. Since $\hat{\boldsymbol{r}}$ vary in time and r is constant, we only need to take the derivative of the former:

$$\frac{d\boldsymbol{r}}{dt} = r\frac{d\hat{\boldsymbol{r}}}{dt} = \omega r\left[-\sin(\omega t)\,\hat{\boldsymbol{x}} + \cos(\omega t)\,\hat{\boldsymbol{y}}\right].$$

b. Let us begin with the expression inside the brackets: $\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt}$, first calculating the time derivative

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= r\frac{d}{dt}\left(\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}\right)\\ &= \omega r\left(-\sin(\omega t)\,\hat{\boldsymbol{x}} + \cos(\omega t)\,\hat{\boldsymbol{y}}\right).\end{aligned}$$

Note that the units of the new vector are $[m/s]$ thus it describes a Using the component representation of the cross product

$$\begin{aligned}\boldsymbol{A} \times \boldsymbol{B} \equiv &\left(A_2B_3 - A_3B_2\right)\hat{\boldsymbol{i}}\\ &+ \left(A_3B_1 - A_1B_3\right)\hat{\boldsymbol{j}}\\ &+ \left(A_1B_2 - A_2B_1\right)\hat{\boldsymbol{k}}.\end{aligned}$$

we find

$$\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt} = \omega r^2\left[\cos^2(\omega t) + \sin^2(\omega t)\right]\hat{\boldsymbol{z}} = \omega r^2\hat{\boldsymbol{z}},$$

note that this relation is the definition of angular velocity vector of the particle: $\omega = \left(\boldsymbol{r} \times \boldsymbol{v}\right)/r^2$.
Now we need to take the time derivative this expression, but since it depends only on constants (ω, r and $\hat{\boldsymbol{z}}$) it vanishes. This also means that the time derivative of the angular velocity is zero,

$$\frac{d\omega}{dt} = 0,$$

the particle moves with a constant angular velocity ω.

2 Angular Location of a Point

The angular location of a point on the edge of a wheel is described by:

$$\phi = At + Bt^2 + Ct^3$$

Where, $A = 4\frac{rad}{sec}$.
$B = -3\frac{rad}{sec^2}$,
and $C = 1\frac{rad}{sec^3}$.

a. Find the angular velocity at $t = 2s$ and at $t = 4s$.

b. Find the average angular acceleration for the time duration Δt that begins at $t = 2s$ and ends at $t = 4s$.

c. Find the momentary angular acceleration at the beginning of Δt and at the end.

Solution:

a.

$$\dot{\phi} = A + 2Bt + 3Ct^2$$

$\dot{\phi}(2s) = A + 4B + 12C = 4\frac{rad}{sec}$,
and $\dot{\phi}(4s) = A + 8B + 48C = 28\frac{rad}{sec}$.

b.

$$\overline{\alpha} = \frac{\Delta\dot{\phi}}{\Delta t} = \frac{28\frac{rad}{sec} - 4\frac{rad}{sec}}{2s} = 12\frac{rad}{sec^2}$$

c.

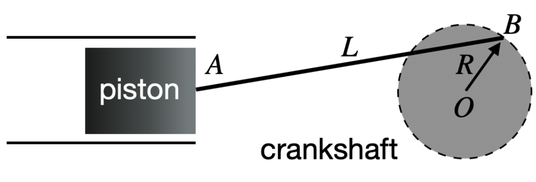
$$\alpha(t) = \frac{d\dot{\phi}}{dt} = 2B + 6Ct$$

$\alpha(2s) = 2B + 12C = 6\frac{rad}{sec^2}$,
and $\alpha(4s) = 2B + 24C = 18\frac{rad}{sec^2}$.

3 Piston

A piston, connected to a wheel with radius by a connecting rod with length L (AB), is free to move along the x axis. The wheel is rotating f times per second.

1. What is the location of point A as a function of time?
2. What is the velocity of point A as a function of time?
3. What is the acceleration of point A as a function of time?



Solution:

1. The position of point B with respect to point O can be written as

$$\vec{OB} = R\left(\cos\varphi, \sin\varphi\right),$$

where $\varphi = 2\pi ft + \varphi_0 = \omega t + \varphi_0$. We can set $\varphi_0 = 0$ by shifting $t = 0$ to the moment when $\varphi = 0$, thus

$$\vec{AB} = \vec{OB} - \vec{OA} = \left(R\cos\varphi + x, R\sin\varphi\right).$$

We also know that

$$L = \left|\vec{AB}\right| = \sqrt{\left(R\cos\varphi + x\right)^2 + R^2\sin^2\varphi}$$
$$= \sqrt{R^2 + 2xR\cos\varphi + x^2},$$

which we solve for x

$$x^2+2xR\cos\varphi+R^2-L^2=0 \quad \rightarrow \quad x=-R\cos\varphi\pm\sqrt{R^2\cos^2\varphi-(R^2-L^2)}=\sqrt{L^2-R^2\sin^2\varphi}-R\cos\varphi.$$

Note that we defined $\vec{OA}=-x\hat{\boldsymbol{x}}$, therefore the relevant solution is with a plus sign ($\varphi=0$ gives us $x=L-R$),

$$\boldsymbol{A}=\left(R\cos(\omega t)-\sqrt{L^2-R^2\sin^2(\omega t)},0\right).$$

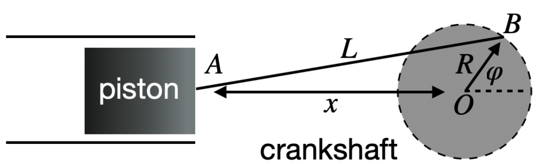
2. The velocity is simply $\dot{\boldsymbol{A}}$,

$$\dot{\boldsymbol{A}}=\left(\frac{\omega R^2\sin(\omega t)\cos(\omega t)}{\sqrt{L^2-R^2\sin^2(\omega t)}}-\omega R\sin(\omega t),0\right)=\left(\frac{\omega R^2\sin(2\omega t)}{2\sqrt{L^2-R^2\sin^2(\omega t)}}-\omega R\sin(\omega t),0\right),$$

where we used $\varphi(t)=\omega t$.

3. The acceleration is $\ddot{\boldsymbol{A}}$,

$$\ddot{\boldsymbol{A}}=\left(\frac{\omega^2R^2\cos(2\omega t)}{\sqrt{L^2-R^2\sin^2(\omega t)}}+\frac{\omega^2R^4\sin^2(2\omega t)}{4\left(L^2-R^2\sin^2(\omega t)\right)^{3/2}}-\omega^2R\cos(\omega t),0\right).$$



4 Acceleration of a Bead on a Spoke

A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t=0$. The angular position of the spoke is given by $\theta=\omega t$, where ω is a constant. Find the velocity and acceleration.

Solution:

The expression for velocity in polar coordinates is

$$\boldsymbol{v}=\dot{\boldsymbol{r}}=\dot{r}\hat{\boldsymbol{r}}+r\frac{d}{dt}\hat{\boldsymbol{r}}=\dot{r}\hat{\boldsymbol{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}},$$

where we can substitute $\dot{r}=u$, $\dot{\theta}=\omega$ and (integrating over the former) $r=ut$, to find $\boldsymbol{v}=u\hat{\boldsymbol{r}}+\omega ut\,\hat{\boldsymbol{\theta}}$. The time derivatives of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ can be derived easily using the cartesian coordinate system or geometry.
The acceleration is

$$\boldsymbol{a}=\dot{\boldsymbol{v}}=\ddot{r}\hat{\boldsymbol{r}}+\dot{r}\frac{d}{dt}\hat{\boldsymbol{r}}+\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}+r\ddot{\theta}\hat{\boldsymbol{\theta}}+r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}=2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}-r\dot{\theta}^2\hat{\boldsymbol{r}},$$

thus $\boldsymbol{a}=-r\omega^2\hat{\boldsymbol{r}}+2u\omega\hat{\theta}$.

5 Motion on a Curve

A particle moves on a plane so that its tangential acceleration $a_t=\alpha$ and its normal acceleration $a_n=\beta t^4$, where α and β are positive constants.
At $t=0$ the particle was at rest.
Find the radius of curvature R (given by $R\equiv\frac{|\boldsymbol{v}|^2}{a_n}$), and the magnitude of the acceleration $\boldsymbol{a}$ as a function of the curve length s the particle has traveled.

Solution:

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \underbrace{\dot{v}\hat{v}}_{a_t} + \underbrace{v\dot{\hat{v}}}_{a_n}$$

$$a_t = \dot{v} = \frac{dv}{dt}$$
$$\int \alpha\, dt = \int dv$$
$$\alpha t = v$$

the total length is given by:

$$\int ds = \int |v|\, dt \Rightarrow s = \frac{1}{2}\alpha t^2.$$

Now we can write t in terms of s in the following way:

$$t = \sqrt{2\frac{s}{\alpha}}$$

Following the formula for R we get:

$$R = \frac{|\boldsymbol{v}|^2}{a_n} = \frac{\alpha^2 t^2}{\beta t^4} = \frac{\alpha^3}{2\beta s}$$

And for the magnitude of $\boldsymbol{a}$:

$$|\boldsymbol{a}| = \sqrt{a_t^2 + a_n^2} = \sqrt{\alpha^2 + \beta^2 t^8} = \alpha\sqrt{1 + 16\beta^2\alpha^{-6}s^4}.$$

6 Fast Train - Bonus

Average velocity of a fast train is about $v = 200$ m/s. The standards require that the perpendicular acceleration will not exceed $a = 0.1g$, in order to keep the passengers comfortable.

1. Find the minimal radius of curvature of the train, assuming its average velocity.

2. What would be the maximal velocity of the train if it passes a curve with a radius of $R_0 = 1$ km?

Definition: Curvature radius is defined as the change of the trajectory with respect to the angle: $R = ds/d\varphi$. This quantity can be interpreted as the radius of a simple circular motion of an object with velocity v and angular velocity ω, since $R = \frac{ds}{dt}/\frac{d\varphi}{dt} = v/\omega$.
Solution:

1. Minimal radius of curvature corresponds to the maximal acceleration (for a constant velocity). Using the relation between radial acceleration and tangent velocity we find

$$a = \frac{v^2}{R} \quad \to \quad R = v^2/a = 40\text{km}.$$

2. Using the same relation we find

$$v = \sqrt{R_0 a} \simeq 32 \text{ m/s}.$$

7 Electron in a Magnetic Field - Bonus

An electron moving in the $\hat{\boldsymbol{x}}$ direction with a velocity V when it enters a magnetic field $\boldsymbol{B}$ that starting at $x = 0$ and ends at $x = D$.
The entire electron motion is on the $x - y$ plane.
The magnetic field causes the electron to feel acceleration in the direction perpendicular to the direction of motion with a magnitude of $a_r = \frac{eB}{m_e}V$.

1. Find Δy for the electron's motion in the magnetic field.

2. Find the angle between the exit direction of the electron and the x axis.

Solution:

The electron has only radial (vertical) acceleration, so its magnitude does not change.
Remember, the magnitude of the velocity varies only due to tangential acceleration.
The magnitude of the radial acceleration is:

$$a_r = \frac{eB}{m_e}V.$$

Since the magnitude of the velocity V does not change, this acceleration does not change, i.e. the radius of curvature does not Changes, and the body moves in a circular motion.
The radius is given by:

$$a_r = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{a_r} = \frac{V^2}{\frac{eB}{m_e}V} = \frac{m_e V}{eB}$$

If $R < D$ the electron makes a half of a circle and goes back out of the magnetic field, so its angle is π and Δy is simply $2R$.
If $R > D$ the electron will perform only an arc out of a circle.
Calculating the angle using geometry and we get:

$$\sin\theta = \frac{R}{D} = \frac{1}{D}\frac{m_e}{eB}V$$

Finding Δy using Pythagorean theorem:

$$D^2 + (R - y)^2 = R^2$$

$$y = R - \sqrt{R^2 - D^2}$$