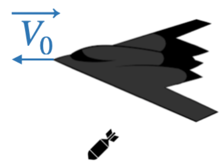
HW 5

1 B-2 Bomber

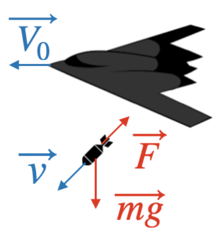
A B-2 bomber is flying with velocity V_0 in the direction of the x axis at high altitude h. At $t = 0$ the bomber drops a bomb with mass m, which experience a velocity dependent friction force of the form $\boldsymbol{F} = -\gamma \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the bomb.



1. Explain the minus sign in the expression for the friction force.
2. Find the velocity of the bomb as a function of time.
3. Find the position of the bomb as a function of time, given $x(t = 0) = 0$.

Solution:

1. The minus sign in the expression of the force denotes that it acts in an opposite direction to that of the velocity.



2. In order to find the expression for the velocity we solve the force equation on the bomb

$$-\gamma \boldsymbol{v} - mg\,\hat{\boldsymbol{y}} = m\dot{\boldsymbol{v}} \quad \rightarrow \quad \dot{\boldsymbol{v}} + \frac{\gamma}{m}\boldsymbol{v} + g\,\hat{\boldsymbol{y}} = 0.$$

Since the gravitational force acts only in the y direction, it will be easier to solve the equation for each direction separately

$$x: \quad \dot{v}_x + \frac{\gamma}{m} v_x = 0$$
$$y: \quad \dot{v}_y + \frac{\gamma}{m} v_y = -g.$$

The x axis solution is straightforward, $v_x(t)$ is a function which return to itself, up to a constant factor, when taking its derivative. Such function is an exponential function

$$v_x = Ae^{\lambda t},$$

plugging it into the equation for the x axis we find

$$A\lambda e^{\lambda t} + \frac{\gamma}{m} A e^{\lambda t} = 0 \quad \to \quad \lambda = -\frac{\gamma}{m}.$$

Whereas the constant A is determined by the initial condition $v_x(t=0) = V_0$

$$v_x(t) = V_0 e^{-\frac{\gamma}{m}t}.$$

While the y axis has a similar solution only with an additional constant factor to account for the constant g in the equation

$$v_y = Be^{\kappa t} + C,$$

so that when we plug it into the equation we find

$$B\lambda e^{\kappa t} + \frac{\gamma}{m} B e^{\kappa t} + \frac{\gamma}{m} C = -g.$$

We see that in order for the time-dependent expression to cancel, we must set $\kappa = \lambda = -\gamma/m$, leaving the constant term $C = -mg/\gamma$. Therefore

$$v_y = Be^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma},$$

where, again, B is set by the initial condition $v_y(t=0) = 0$, which leads to $B = \frac{mg}{\gamma}$, thus

$$v_y = \frac{mg}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right).$$

We should check that the units are correct, the units for γ are force/velocity therefore

$$\left[\frac{mg}{\gamma}\right] = [\text{velocity}].$$

Writing the vector expression for the velocity yields

$$\boldsymbol{v}(t) = V_0 e^{-\frac{\gamma}{m}t}\,\hat{\boldsymbol{x}} + \frac{mg}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right)\hat{\boldsymbol{y}}.$$

We should also preform a sanity check: as $t \to \infty$ we find that the velocity has no x component, which is reasonable since the friction force makes sure the velocity attenuates, while the velocity on the y axis reaches a maximal value, which corresponds to the equilibrium state where the net force is zero $mg = \gamma v_{max}$.

3. Once we obtained an expression for $\boldsymbol{v}(t)$, all we need to do in order to find the position vector is integrate over the velocity vector

$$\begin{aligned}
\boldsymbol{r}(t) &= \int \boldsymbol{v}(t)\,dt \\
&= \int \left[V_0 e^{-\frac{\gamma}{m}t}\,\hat{\boldsymbol{x}} + \frac{mg}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right)\hat{\boldsymbol{y}}\right] dt \\
&= -\frac{mV_0}{\gamma} e^{-\frac{\gamma}{m}t}\,\hat{\boldsymbol{x}} - \frac{mg}{\gamma}\left(\frac{m}{\gamma} e^{-\frac{\gamma}{m}t} + t\right)\hat{\boldsymbol{y}} + \boldsymbol{r}_0,
\end{aligned}$$

where $\boldsymbol{r}_0$ is determined from the initial conditions $x(t=0) = 0$ and $y(t=0) = h$,

$$-\frac{mV_0}{\gamma}\hat{\boldsymbol{x}} - \frac{m^2}{\gamma^2} g\hat{\boldsymbol{y}} + \boldsymbol{r}_0 = h\hat{\boldsymbol{y}} \quad \to \quad \boldsymbol{r}_0 = \frac{mV_0}{\gamma}\hat{\boldsymbol{x}} + \left(h + \frac{m^2}{\gamma^2} g\right)\hat{\boldsymbol{y}}.$$

Therefore

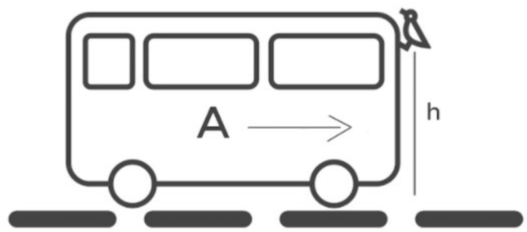
$$\boldsymbol{r}(t) = \frac{mV_0}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right)\hat{\boldsymbol{x}} + \frac{mg}{\gamma}\left[\frac{h\gamma}{mg} + \frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right) - t\right]\hat{\boldsymbol{y}}.$$

2 Bus and Bird

A bus accelerates to the right with acceleration A.

A bird with mass m sat on the front window of the bus at height h from the road and will not fly off for the rest of this question.
There is a friction coefficient between the bird and the bus $\mu_s = \mu_k = \mu$.

1. What is the minimal acceleration A_{min} for which the bird will not slip?
2.If $A = \frac{A_{min}}{2}$, what will be the acceleration of the bird relative to the bus and relative to the ground? Where will the bird get to the ground?



Solution:

1.Writing the equations of motion:
In the horizontal direction the bird is moving with the bus:

$$N = mA$$

In the vertical direction, the birds 'wants' to slide down so the frictional force directed up:

$$f - mg = 0$$

and $f \le \mu N = \mu m A$

$$mg = f \le \mu m A$$

so

$$A_{min} = \frac{g}{\mu}$$

2.When $A = \frac{g}{2\mu}$ the vertical accelerations is $\mu A - g = -\frac{g}{2}$
So in the frame moving with the bus

$$\vec{a}_{Bird} = \left(0, -\frac{g}{2}\right)$$

And in the frame of the ground

$$\vec{a}_{Bird} = \left(A, -\frac{g}{2}\right)$$

The vertical displacement of the bird as function of time is:

$$-h = \Delta y = -\frac{1}{2}\frac{g}{2}t^2$$

So the bird get to the ground at

$$t_{ground} = 2\sqrt{\frac{h}{g}}$$

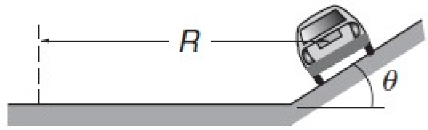
The horizontal distance the bus passed until t_{ground} is given by

$$\Delta x = \frac{1}{2}At^2 = \frac{g}{2\mu}4\frac{h}{g} = \frac{2}{\mu}h$$

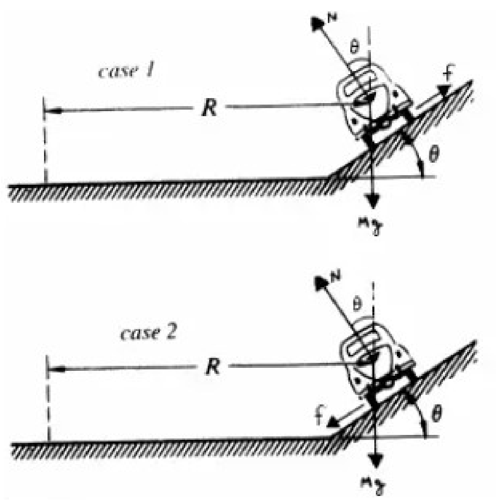
And the bird is on the ground in a distance $\frac{2h}{\mu}$ to the right of the location at the beginning of the motion.

3 Turning Car

A car of mass M enters a turn whose radius is R.
The road is banked at angle θ, and the coefficient of friction between the wheels and the road is μ.
Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.



Solution:



There are two cases, as the sketches indicates.
Keep in mind that the friction force is opposed to the direction of motion.
In case 1, the car will tend to slide down the slope if it moving too slowly, so the friction force f is outwards as shown.
In case 2, the car will tend to slide up the slope if it moving too fast, so f is inwards.
Case 1:
Horizontal equation of motion:

$$Ma_C = M\frac{v^2}{R} = N\sin\theta - f\cos\theta$$

The maximum friction force is μN

$$M\frac{v^2}{R} \geq N\left(\sin\theta - \mu\cos\theta\right)$$

$$M\frac{v_{min}^2}{R} = N\left(\sin\theta - \mu\cos\theta\right)$$

There is no vertical acceleration if the car is not sliding, so the vertical equation of motion is

$$N\cos\theta + f\sin\theta - Mg = 0$$

In the limit where $f = \mu N$

$$Mg = N\left(\cos\theta + \mu\sin\theta\right)$$

and we get

$$\frac{v_{min}^2}{R} = g\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} \Rightarrow v_{min} = \sqrt{Rg\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right)}$$

Case 2:
Proceeding as before,

$$M\frac{v^2}{R} \leq N\sin\theta + f\cos\theta$$

$$M\frac{v^2}{R} \leq N\left(\sin\theta + \mu\cos\theta\right)$$

Vertical equation of motion:

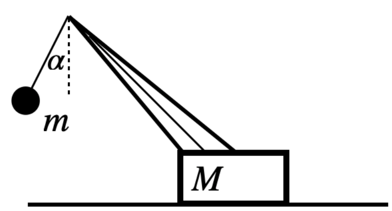
$$N\cos\theta - f\sin\theta - Mg = 0$$

$$Mg = N\left(\cos\theta - \mu\sin\theta\right)$$

$$v_{max} = \sqrt{Rg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)}.$$

4 Wrecking Ball

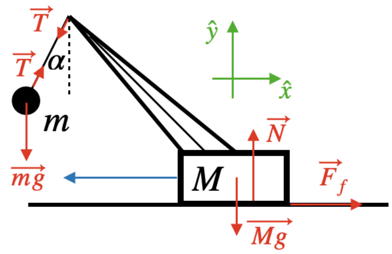
A wrecking ball crane with mass M is moving on a horizontal plane with coefficient of kinetic friction μ_k. During the motion, the wrecking ball, which is hanged from the crane, keeps a constant angle α with the vertical axis (see figure).
note: Treat the masses in the problem as point masses (cannot rotate around themselves).



1. What is the expression for the tension of the cable?
2. What is the expression for the normal force on the crane M?
3. What is the expression for the acceleration of the crane M?
4. What is the expression for the friction force and the coefficient of kinetic friction μ_k?
5. What is the direction of the motion?

Solution:

Since there are no drive forces (engine or any force that generates acceleration in the direction of motion) it is safe to assume that the acceleration would be in the direction of the friction force (opposite to the direction of the motion), which would cause the wrecking ball to tilt in the opposite direction (i.e. opposite to the friction force). Therefore we assume that the motion is to the left direction (if we are wrong we will get a minus sign in some of our expressions).



1. Let us first write down the equations of motion for the wrecking ball, since the angle α is constant and the crane does not move in the y direction, we know that the acceleration of the wrecking ball is only in the x direction, thus
$$T\sin\alpha\hat{\boldsymbol{x}} + (T\cos\alpha - mg)\,\hat{\boldsymbol{y}} = ma\,\hat{\boldsymbol{x}},$$
where a is the acceleration of the wrecking ball. Solving the equation for the y component we find
$$T = \frac{mg}{\cos\alpha}.$$

2. Now, let us write the equations of motion for the crane
$$(F_f - T\sin\alpha)\,\hat{\boldsymbol{x}} + (N - Mg - T\cos\alpha)\,\hat{\boldsymbol{y}} = Ma\hat{\boldsymbol{x}},$$
where we used a as the acceleration of the crane, again since the angle α is constant - we understand that the velocities of the wrecking ball and the crane must be the same and therefore also the accelerations of both objects. We find the normal force from the y component of the equation and using the expression we found for T
$$N = (M+m)\,g.$$

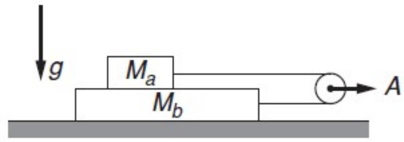
3. Now that we've found the expressions for T, we turn to the x component of the equation for m
$$mg\tan\alpha = ma \quad \rightarrow \quad a = g\tan\alpha.$$

4. Doing the same for M we find
$$F_f - mg\tan\alpha = Mg\tan\alpha \quad \rightarrow \quad F_f = g\,(M+m)\tan\alpha.$$

5. Recalling that $F_f = N\mu_k$ we isolate μ_k to find
$$N\mu_k = g\,(M+m)\tan\alpha \quad \rightarrow \quad \cancel{(M+m)}\,g\mu_k = g\,\cancel{(M+m)}\tan\alpha \quad \rightarrow \quad \mu_k = \tan\alpha,$$
thus the angle α can be used to deduce the coefficient of kinetic friction.

5 Stacked Blocks and Pulley

Mass M_a lies on top of mass M_b, as shown. Assume $M_b > M_a$.
The two blocks are pulled from rest by a massless rope passing over a pulley. The pulley is accelerated at rate A. Block M_b slides on the table without friction, but there is a constant friction force f between M_a and M_b due to their relative motion. Find the tension in the rope.



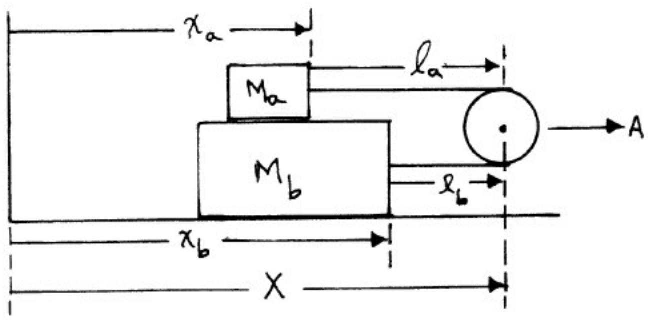
Solution:

We define the coordinates of the objects in this problem - we choose some point to the left of the two masses to be $x = 0$, the location of mass M_a denoted by x_a, the location of mass M_b denoted by x_b, and the location of the pulley by X.
The forces equation for each mass:

$$M_a \ddot{x_a} = T - f$$

$$M_b \ddot{x_b} = T + f$$

Since $M_b > M_a$ and M_a is on top of M_b it is safe to assume M_a is moving faster to the right (The friction force f is opposing to the relative motion of the masses).



We got 2 eqn but 3 variables to find, to solve we need a constraint. Using that the rope has a fixed length

$$(X - x_a) + (X - x_b) = constant \left(\frac{d^2}{dt^2} () \right)$$

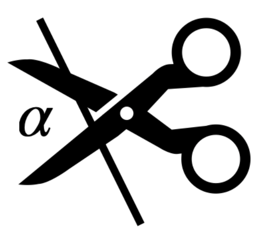
$$2\ddot{X} - \ddot{x_a} - \ddot{x_b} = 0 \Rightarrow \ddot{x_a} = 2A - \ddot{x_b}$$

and we get

$$\frac{T - f}{M_a} = 2A - \frac{T + f}{M_b} \Rightarrow \underline{T = 2A\left(\frac{M_a M_b}{M_a + M_b}\right) + f\left(\frac{M_b - M_a}{M_a + M_b}\right)}$$

6 Cutting Wire - Bonus

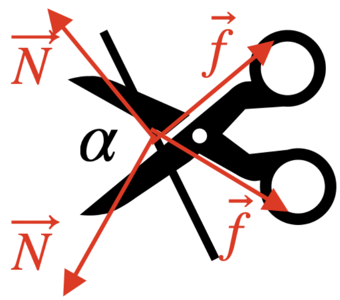
Someone is trying to cut a metallic wire using scissors. The wire is not connected to anything and it slides out until the angle between the blades is α. Only then do the scissors start cutting the wire. You may ignore gravity.



1. Draw the forces acting on the wire.
2. What is the coefficient of friction between the blades and the wire?
3. Now we introduce gravity into the force balance. Show that α remains unchanged when the scissors are held as in the figure.
4. Show that if the scissors are rotated upwards about their screw, by an angle θ, then α changes such that such that $\mu = \mu_0 + \Delta\mu$, where μ_0 is the coefficient of friction found in (2), and $\Delta\mu \equiv -\frac{mg\sin\theta}{F\cos(a/2)}$. Here F is the total force applied to the scissors. Will the scissors cut earlier or later when pointed upwards (i.e. $\theta > 0$)?

Solution:

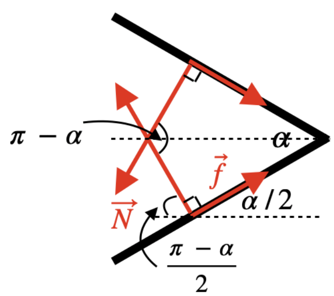
1. Neglecting gravity, the only forces acting on the wire are the normal forces from the blades:



2. Since the wire stops sliding exactly at angle α, we can write the equations of motion as

$$x: \quad -2N_x + 2f_x = 0,$$

while the y component of the total force is zero as the Ns and fs cancel out. The angles are as follows



therefore, the equation for the x component reads

$$N\sin\frac{\alpha}{2} - N\mu_0\cos\frac{\alpha}{2} = 0 \quad \rightarrow \quad \mu_0 = \tan\frac{\alpha}{2}.$$

3. Introducing gravity changes the normal, and therefore the friction, from the lower blade, thus

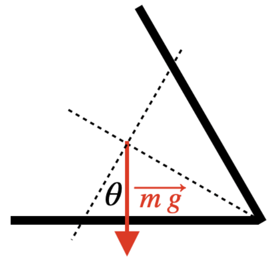
$$x: \quad -N'_x - N_x + f'_x + f_x = 0,$$

or

$$-\cancel{(N'+N)}\sin\frac{\alpha}{2} + \mu_0\cancel{(N'+N)}\cos\frac{\alpha}{2} = 0 \quad \rightarrow \quad \mu_0 = \tan\frac{\alpha}{2}.$$

The result is unchanged.

4. We rotate our coordinate system along with the scissors so that the gravitational force has a component along the new x axis



$$x: \quad -N'_x - N_x + f'_x + f_x + mg\sin\theta = 0,$$

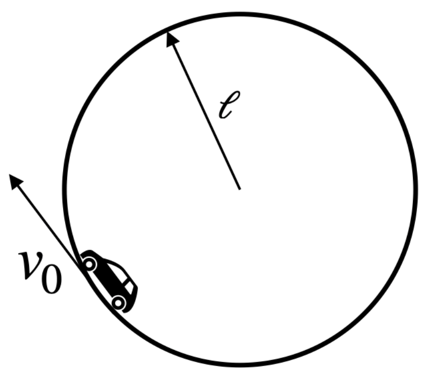
Plugging in the expression for the friction force and the components of the vectors reads

$$-(N'+N)\sin\frac{\alpha}{2} + \mu(N'+N)\cos\frac{\alpha}{2} + mg\sin\theta = 0 \quad \to \quad \mu = \tan\frac{\alpha}{2} - \frac{mg}{(N'+N)}\frac{\sin\theta}{\cos\frac{\alpha}{2}},$$

where $N'+N$ is the total force applied on the scissors. We find $\mu < \mu_0$, if $\theta > 0$, therefore cutting with the same angle α would correspond to using a blunter scissors. Thus, if we fix μ, the scissors will cut earlier.

7 Toy Car - Bonus

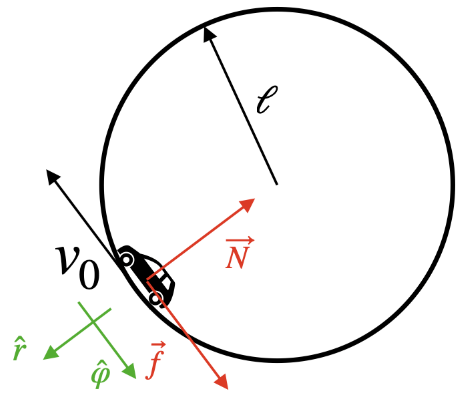
A toy car is fixed to a track on a ring with radius ℓ, which is stationed horizontally on a frictionless table, as shown in the figure. at $t = 0$, as the car is traveling along the inner side of the ring with velocity v_0, its tires are punctured so its bottom slides on the ring with coefficient of kinetic friction μ.



1. Find the velocity of the car at time t.
2. Find the distance the car travels after time t.

Solution:

1. Since there is no friction with the table, the vertical axis plays no role in the motion of the car, as the normal from the table cancels the gravitational force.
   The only relevant forces are the normal from the ring (radial force) and the friction with the ring (tangent force):



so that the equations of motion are

$$r: \quad -N = -m\ell\dot{\varphi}^2,$$
$$\varphi: \quad f = m\ell\ddot{\varphi}.$$

Since $f = N\mu$ we can combine both equations into

$$\ddot{\varphi} = \mu\dot{\varphi}^2 \quad \rightarrow \quad \frac{d\dot{\varphi}}{\dot{\varphi}^2} = \mu dt$$

so that

$$-\dot{\varphi}^{-1} = \mu t + C.$$

Using initial conditions $\dot{\varphi}(t=0) = -v_0/\ell$ we find $C = \ell/v_0$, thus

$$\dot{\varphi}(t) = -\frac{1}{\mu t + \ell/v_0} \quad \rightarrow \quad v(t) = -\frac{v_0}{\mu v_0 t/\ell + 1}.$$

Sanity check shows that as $t \to \infty$ the velocity goes to zero, as it should.

2. Integrating $|v(t)|$ gives the total distance travelled from $t = 0$ to t

$$\begin{aligned} S(t) &= \int_0^t \frac{v_0}{\mu v_0 t'/\ell + 1} dt' \\ &= \frac{\ell}{\mu} \ln(\ell + \mu v_0 t)\Big|_0^t \\ &= \frac{\ell}{\mu} \ln\left(1 + \frac{\mu v_0}{\ell} t\right). \end{aligned}$$

If we take $\mu = 0$ we find

$$\lim_{\mu \to 0} \frac{\ell}{\mu} \ln\left(1 + \frac{\mu v_0}{\ell} t\right) = v_0 t,$$

which is the result with zero acceleration, as it should be for zero friction.