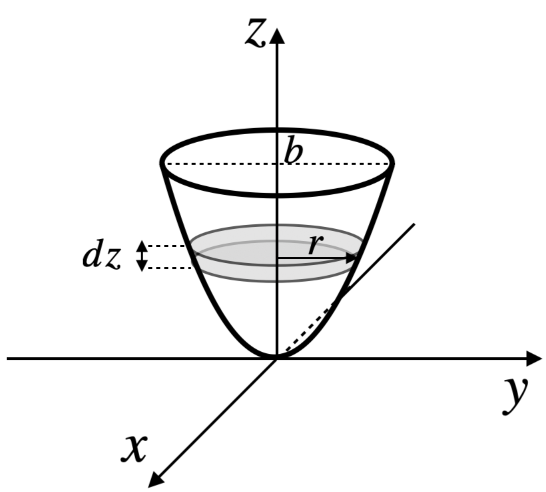
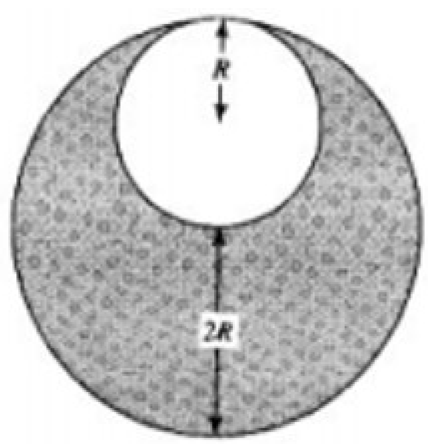
HW 6

1 Center of mass calculations

Find the center of mass for the following bodies:

1. A paraboloid $z = a\left(x^2 + y^2\right)$ between $z = 0$ and $z = b$ with uniform density $\rho = \rho_0 \frac{z}{a}$.
2. A paraboloid $z = a\left(x^2 + y^2\right)$ between $z = 0$ and $z = b$ with uniform density $\rho = \rho_0 \frac{a}{z}$.
3. A disk of radius $2R$ with a uniform surface density σ which has a circular hole of radius R at distance R from the center of the first circle.





Solution:

Generally speaking,

$$\vec{R}_{CM} = \frac{1}{M_{tot}} \int \vec{r}\, dm$$

where,

$$M_{tot} = \int dm$$

and

$$dm = \begin{cases} \lambda dl & 1D \\ \sigma dA & 2D \\ \rho dV & 3D \end{cases}$$

1. We exploit the symmetry around the $\hat{z}$ axis in this problem and switch to cylindrical coordinates in which the paraboloid described by

$$z = ar^2$$

The integral limits are $z : 0 \to b$, $r : 0 \to \sqrt{\frac{z}{a}}$ and $\theta : 0 \to 2\pi$

Writing the integrals, don't forget the Jacobin! $dxdydz \to rdrd\theta dz$

$$M_{tot} = \int_0^{2\pi}\int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} \rho(z) r dr z d\theta$$

We have to first integrate by r and then by z because the limits of r depend on z.

$$M_{tot} = 2\pi \int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} \left(\rho_0 \frac{z}{a}\right) r dr z = \frac{2\pi\rho_0}{a}\int_0^{b}\left[\frac{r^2}{2}\right]_0^{\sqrt{\frac{z}{a}}} z dz =$$

$$= \frac{\pi\rho_0}{a^2}\int_0^{b} z^2 dz = \frac{\pi\rho_0 b^3}{3a^2}$$

Unit Check: $z = ar^2 \Rightarrow [a] = m^{-1}$,$\rho(z) = \rho_0\frac{z}{a} \Rightarrow [\rho_0] = kg\, m^{-5}$,$[b] = m$

$$kg = [M_{tot}] = \left[\rho_0 \frac{b^3}{a^2}\right] = \frac{kg}{m^5}\frac{m^3}{m^{-2}} = kg$$

$$\overrightarrow{R}_{CM} = \frac{3a^2}{\pi\rho_0 b^3}\int_0^{2\pi}\int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} \overrightarrow{r}(r,\theta,z)\rho(z) r dr z d\theta$$

where $\overrightarrow{r}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

$$\overrightarrow{R}_{CM} = \frac{3a^2}{\pi\rho_0 b^3}\int_0^{2\pi}\int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} (r\cos\theta, r\sin\theta, z)\,\frac{\rho_0 z}{a} r dr z d\theta$$

Note that $\int_0^{2\pi}\cos\theta = \int_0^{2\pi}\sin\theta = 0$.

$$R_{CM,z} = \frac{3a}{\pi b^3}\int_0^{2\pi}\int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} z^2 r dr z d\theta = \frac{6a}{b^3}\int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} z^2 r dr z =$$

$$= \frac{6a}{b^3}\int_0^{b} z^2\left[\frac{r^2}{2}\right]_0^{\sqrt{\frac{z}{a}}} dz = \frac{3}{b^3}\int_0^{b} z^3 dz = \frac{3}{4}b$$

and we get

$$\overrightarrow{R}_{CM} = \left(0, 0, \frac{3}{4}b\right).$$

2. For $\rho(z) = \rho_0\frac{a}{z}$

$$M_{tot} = 2\pi \int_0^{b}\int_0^{\sqrt{\frac{z}{a}}} \left(\rho_0 \frac{a}{z}\right) r dr z = 2\pi\rho_0 a\int_0^{b}\frac{1}{z}\left[\frac{r^2}{2}\right]_0^{\sqrt{\frac{z}{a}}} z dz =$$

$$= \pi\rho_0\int_0^{b} dz = \pi\rho_0 b$$

Unit Check: $[a] = m^{-1}$,$\rho(z) = \rho_0 \frac{a}{z} \Rightarrow [\rho_0] = kg\, m^{-1}$,$[b] = m$

$$kg = [M_{tot}] = [\rho_0 b] = \frac{kg}{m} m = kg$$

$$\vec{R}_{CM} = \frac{1}{\pi \rho_0 b} \int_0^{2\pi} \int_0^{b} \int_0^{\sqrt{\frac{z}{a}}} \vec{r}(r, \theta, z)\rho(z) r dr dz d\theta$$

$$R_{CM,z} = \frac{1}{\pi \rho_0 b} \int_0^{2\pi} \int_0^{b} \int_0^{\sqrt{\frac{z}{a}}} z \rho_0 \frac{a}{z} r dr dz d\theta = \frac{2a}{b} \int_0^{b} \int_0^{\sqrt{\frac{z}{a}}} r dr z =$$

$$= \frac{2a}{b} \int_0^{b} \left[\frac{r^2}{2}\right]_0^{\sqrt{\frac{z}{a}}} dz = \frac{1}{b} \int_0^{b} z dz = \frac{b}{2}$$

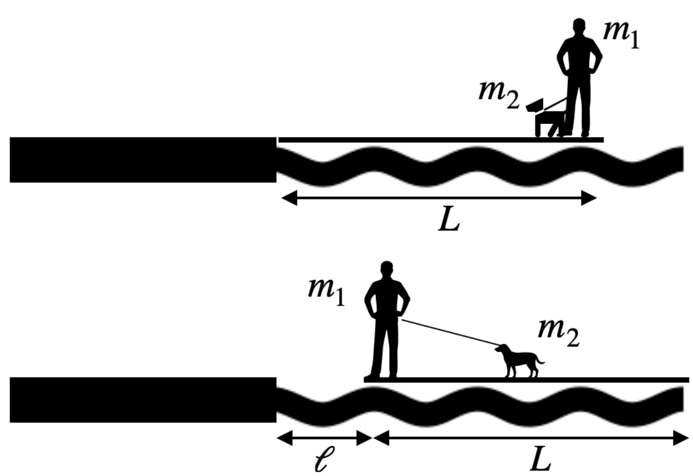
$$\vec{R}_{CM} = \left(0, 0, \frac{1}{2} b\right).$$

3. Use superposition!
   The complicated disc can be seen as made of
   *A full disc of radius $2R$ with surface mass density $+\sigma$ (mass $M = 4\pi R^2 \sigma$) centered at the origin.
   *A full disc of radius R with a surface mass density $-\sigma$ (mass $m = -\pi R^2 \sigma$) centered at $y = R$.
   The y coordinate of the center of mass: (x coordinate is zero from symmetry)

$$Y_{CM} = \frac{M \cdot 0 + m \cdot R}{M + m} = -\frac{\pi R^2 \sigma}{3\pi R^2 \sigma} R = -\frac{R}{3}.$$

2 Man, Dog and a Raft

A person with mass m_1 and his dog m_2, on a leash, are standing at rest together on the right edge of a raft with length L and mass M, floating in the calm water of the Kineret. the person tries to leave the raft and walks towards the dock (at the left side of the raft), with constant velocity v_1. The dog, initially follows his master but eventually grasps that this is pointless and halts at the center of the raft. The person reaches the left edge of the raft and, luckily, stopped by the leash of his dog, so that the person stands at the left edge of the raft, whereas the dog is at its center.
Assuming no frictional forces between the raft and the water, what is the distance ℓ that the raft travelled by the time the person was halted?



Solution:

Let us define the origin at the dock $x = 0$. Since no external forces act on the system of the person+dog+raft, the center of mass (CM) is conserved. Before the motion it is

$$x_{CM}(t_i) = \frac{(m_1 + m_2)L + ML/2}{m_1 + m_2 + M},$$

whereas at the end of the motion, it is

$$x_{CM}(t_f) = \frac{m_1\ell + (m_2 + M)(\ell + L/2)}{m_1 + m_2 + M}.$$

Therefore

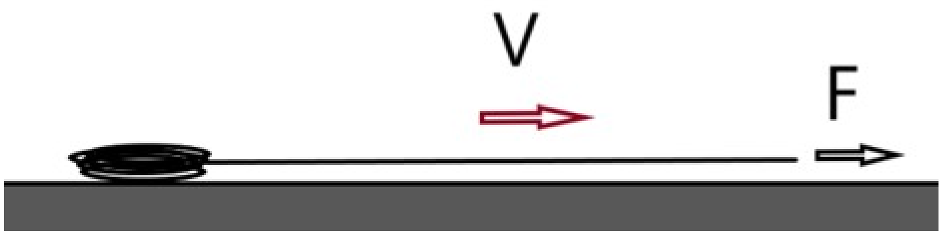
$$x_{CM}(t_f) = x_{CM}(t_i) \quad \rightarrow \quad \ell = \frac{(m_1 + m_2)L + ML/2 - (m_2 + M)L/2}{m_1 + m_2 + M}$$

or simply

$$\ell = \frac{m_1 + m_2/2}{m_1 + m_2 + M}L.$$

3 Rope on table

A long rope rolled on a horizontal table is pulled from the end at a constant horizontal velocity $\boldsymbol{V}$. There is no friction between the table and the rope. The rope mass per unit length is λ.



1. Find the magnitude of the force $\boldsymbol{F}$pulling the rope that satisfies this motion.
2. If the force was pulling upwards the rope with a constant velocity $\boldsymbol{V}$, would its magnitude change?
   (Back to the horizontal motion)
   When the entire length of the rope l is stretched the force stops.
   A small part of the rope l_0 hung from the edge of the table.
   Initially the rope is at rest.
3. Find an equation $y(t)$, the length of the rope that crossed the edge of the table at time t.

Solution:

1. The change in the momentum on the x direction is

$$P_x(t) = mV$$

$$P_x(t + \Delta t) = (m + \Delta m)V$$

$$\Delta P_x = \Delta m V = F\Delta t$$

Divide by Δt and taking it to be infinitesimally small.

$$\frac{dP_x}{dt} = F = \frac{dm}{dt}V = \frac{d(\lambda L)}{dt}V = \lambda\frac{dL}{dt}V = \lambda V^2$$

so

$$F = \lambda V^2.$$

2. The change in the momentum on the y direction is

$$P_y(t) = mV$$
$$P_y(t+\Delta t) = (m+\Delta m)\,V$$
$$\frac{dP_y}{dt} = F - mg = \frac{dm}{dt}V$$
$$m = \lambda V t$$
$$F = \lambda V(V+gt).$$

3. At time $t=0$, the rope is momentarily at rest $\dot{y}(0)=0$, with length $y(0)=l_0$ hanging.
Note that if the hanging part is falling with a velocity $-u\hat{y}$ then the rope that is on the table moving along the table with velocity $u\hat{x}$

$$\overrightarrow{P}(t) = \lambda\,(l-y)\,u\hat{x} - \lambda y u\hat{y}$$
$$\overrightarrow{P}(t+\Delta t) = \lambda\,(l-y-\Delta y)\,(u+\Delta u)\,\hat{x} - \lambda\,(y+\Delta y)\,(u+\Delta u)\,\hat{y}$$

We'll ignore terms of $\Delta y\Delta u$, $\Delta y\Delta u \ll u\Delta y$, $y\Delta u$.

$$\overrightarrow{P}(t+\Delta t) = \lambda\,(l\,(u+\Delta u) - y\,(u+\Delta u) - \Delta y u)\,\hat{x} - \lambda\,(yu + y\Delta u + \Delta y u)\,\hat{y}$$
$$\Delta\overrightarrow{P} = \lambda\,((l-y)\,\Delta u - u\Delta y)\,\hat{x} - \lambda\,(y\Delta u + u\Delta y)\,\hat{y}$$

There is only external gravitational force in the $\hat{y}$ direction, so

$$0 = \Delta P_x = \lambda\,((l-y)\,\Delta u - u\Delta y)$$
$$\Delta y = \frac{l-y}{u}\Delta u$$
$$-\lambda y g \Delta t \approx \int_t^{t+\Delta t} -\lambda y g\, dt = \Delta P_y = -\lambda\,(y\Delta u + u\Delta y)$$
$$yg\Delta t = y\Delta u + u\frac{l-y}{u}\Delta u = l\Delta u$$

Divide by Δt and taking it to be infinitesimally small. We get

$$y = \frac{l}{g}\frac{du}{dt} = \frac{l}{g}\ddot{y}$$

The general solution of the differential equation is

$$y(t) = A\exp\left[\sqrt{\frac{g}{l}}t\right] + B\exp\left[-\sqrt{\frac{g}{l}}t\right]$$

Check the solution

$$\dot{y}(t) = \sqrt{\frac{g}{l}}\left(A\exp\left[\sqrt{\frac{g}{l}}t\right] - B\exp\left[-\sqrt{\frac{g}{l}}t\right]\right)$$
$$\ddot{y}(t) = \frac{g}{l}\left(A\exp\left[\sqrt{\frac{g}{l}}t\right] + B\exp\left[-\sqrt{\frac{g}{l}}t\right]\right) = \frac{g}{l}y(t)$$

it works!
Finding A and B using initial condition:

$$y(0) = l_0 \Rightarrow A + B = l_0$$
$$\dot{y}(0) = 0 \Rightarrow A - B = 0$$

Therefore the solution is

$$y(t) = \frac{l_0}{2}\left(\exp\left[\sqrt{\frac{g}{l}}t\right] + \exp\left[-\sqrt{\frac{g}{l}}t\right]\right) = l_0\cosh\left(\sqrt{\frac{g}{l}}t\right).$$

4 Spacecrafts and dust streams

A spacecraft moves in space at a constant velocity $\overrightarrow{v}$.
On its way, the spacecraft encounters a stream of dust particles that connect to the spacecraft's body at a rate $\frac{dm}{dt}$, with the particles having a velocity of $\overrightarrow{u}$ just before they hit the spacecraft.
At time t the spacecraft has a mass $M(t)$.
What force does the engine of the spacecraft need to apply to it so that it can continue to move at a constant velocity?

Solution:

At time t our system consists from the spacecraft and the dust already on in with total mass of $M(t)$ and the dust particles that still moves freely and will hit the spacecraft in period of time Δt with mass ΔM.

$$\overrightarrow{P}(t) = M(t)\overrightarrow{v} + \Delta M\overrightarrow{u}$$

$$\overrightarrow{P}(t+\Delta t) = (M(t) + \Delta M)\,\overrightarrow{v}$$

Using the second law $\frac{d\overrightarrow{P}}{dt} = \overrightarrow{F}$

$$\Delta\overrightarrow{P} = \Delta M\left(\overrightarrow{v} - \overrightarrow{u}\right) = \overrightarrow{F}\Delta t$$

$$\overrightarrow{F} = \frac{dm}{dt}\left(\overrightarrow{v} - \overrightarrow{u}\right).$$

5 Rocket sled

A rocket sled moves along a horizontal plane, and is retarded by a friction force $f_{\text{friction}} = \mu W$, where μ is constant and W is the weight of the sled.
The sled's initial mass is M_0, and its rocket engine expels mass at constant rate $\frac{dM}{dt} \equiv \gamma$; the expelled mass has constant speed v_0 relative to the rocket.
The rocket sled starts from rest and the engine stops when half the sled's total mass is gone.
Find an expression for the maximum speed.

Solution:

At time t, the system consists of mass M moving with speed v, where $M(0) = M_0$.
The momentum $P(t)$ is

$$P(t) = Mv$$

At time $t + \Delta t$, the system (still of mass M) consists of the mass $(M - \Delta M)$ moving with speed $(v + \Delta v)$ and mass ΔM moving with speed $(v - v_0)$. Then

$$\begin{aligned}\Delta P &= P(t+\Delta t) - P(t)\\ &= (M - \Delta M)(v + \Delta v) + \Delta M\,(v - v_0) - Mv\end{aligned}$$

Keeping only 1st order in Δ:

$$\Delta P = M\Delta v - \Delta M v_0$$

In the limit $\Delta t \to 0$

$$\frac{dP}{dt} = M\frac{dv}{dt} - v_0\frac{dM}{dt}$$

The friction force on the sled is $-\mu Mg$.

$$\frac{dP}{dt} = -\mu Mg$$

$$M\frac{dv}{dt} - v_0\frac{dM}{dt} = -\mu M g$$

The fuel burns at consant rate $\frac{dM}{dt} = -\gamma$, so that $M(t) = M_0 - \gamma t$.

$$\frac{dv}{dt} = \frac{v_0\gamma}{M_0 - \gamma t} - \mu g$$

Integrating,

$$v(t) = \int_0^t \left[\frac{v_0\gamma}{M_0 - \gamma t} - \mu g\right] dt = v_0\gamma \int_0^t \frac{dt'}{M_0 - \gamma t'} - \mu g t$$

$u = M_0 - \gamma t \to du = -\gamma dt$:

$$v(t) = v_0\gamma \int_{M_0}^{M_0-\gamma t} -\frac{1}{\gamma}\frac{du}{u} - \mu g t = v_0 \ln\left(\frac{M_0}{M_0 - \gamma t}\right) - \mu g t$$

The rocket engine turns off at time t_f when $M_0 - \gamma t_f = \frac{M_0}{2} \Rightarrow t_f = \frac{M_0}{2\gamma}$
The sled begins to slow for $t > t_f$, so the maximum speed is

$$v(t_f) = v_0 \ln 2 - \mu g \frac{M_0}{2\gamma}.$$

6 Shell - Bonus

A shell with a total mass $M = m_1 + m_2$ is fired from a cannon with initial velocity v_0 and angle θ. At the maximal height of its trajectory the shell splits into two parts m_1 and m_2. Given that both parts hit the ground at the same time, and that m_1 ends up a distance D_1 from the cannon:

1. What is the impact on the shell the moment it is fired?
2. Where does m_2 land?
3. What is the impact on each of the masses at the moment of the splitting?

Solution:

1. The impact on the shell at the moment it is fire is the change in the momentum, thus

$$\Delta \boldsymbol{p} = M\Delta \boldsymbol{v} = M v_0 \left(\cos\theta\, \hat{\boldsymbol{x}} + \sin\theta\, \hat{\boldsymbol{y}}\right).$$

2. Since no external forces acts on the shell, we know that the CM is conserved. since both masses hit the ground at the same time, the CM (i.e. the original shell M) would have hit the ground at the same time. We can find where the CM lands and use this, and the landing position of m_1, in order to find D_2. The CM follow

$$x_{CM}(t) = v_0 \cos\theta\, t,$$
$$y_{CM}(t) = v_0 \sin\theta\, t - \frac{g}{2}t^2,$$

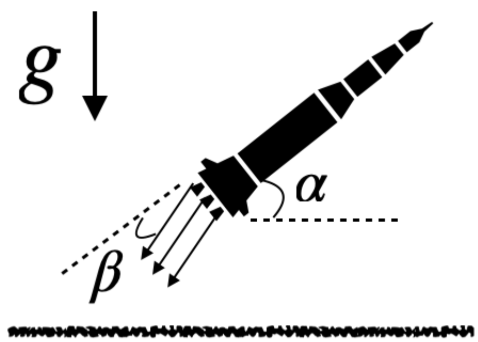
therefore, it hits the ground at $t_{hit} = 2v_0 \sin\theta/g$, which corresponds to $x_{CM}(t_{hit}) = v_0^2 \sin 2\theta/g$. Therefore

$$x_{CM}(t_{hit}) = \frac{m_1 D_1 + m_2 D_2}{m_1 + m_2} \quad \to \quad D_2 = \left(\frac{m_1}{m_2} + 1\right)\frac{v_0^2}{g}\sin(2\theta) - \frac{m_1}{m_2}D_1.$$

3. Assuming that the splitting took negligible amount of time, the impact on each mass is
$$\Delta \boldsymbol{p}_i = m_i \left(v_{x,i} - v_0 \cos\theta \right).$$
The velocities $v_{x,i}$ are found from kinematics: first let us find the time it took to reach maximum height, but that is half the time t_{hit}
$$t_{\text{fall}} = \frac{v_0}{g} \sin\theta.$$
Next we find the velocity of each mass
$$v_{x,i} = \frac{D_i - x_{CM}}{t_{\text{fall}}}.$$

7 Firework Rocket - Bonus

A firework rocket is leaving the ground and travels with a constant angle α, as shown in the figure. The rocket ejects gas so that its mass follows $m(t) = m_0 e^{-\gamma t}$. The gas is ejected with velocity $u > 0$ relative to the rocket. The rockets engine can adjust its angle so that the gas will eject at angle β relative to the directionality of the rocket, as shown in the figure. Due to the small dimensions of the rocket, it is safe to assume that it does not rotate, it is also given that the rocket moves in an inertial frame with constant gravitational acceleration g.



1. What is the angle β that allows the rocket to move in the direction of α?
2. What is the velocity of the rocket as a function of time, observed from the ground, as long as the gas does not run out? Assume that at $t = 0$, the rocket is at rest.

Solution:

In order for the rocket to have velocity only in the direction of α, the acceleration of the rocket $\dot{\boldsymbol{v}}$ must be in this direction alone. The easiest coordinate system to use would be the one with $\hat{\boldsymbol{x}}$ along the direction of movement of the rocket. The mass loss rate of the rocket is $|\dot{m}| = \gamma m(t)$.

1. The y component of the acceleration must be zero, thus
$$dp_y = -mg\cos\alpha = m\cancelto{0}{\dot{v}_y} - |\dot{m}|\, u \sin\beta \quad \rightarrow \quad \sin\beta = \frac{g\cos\alpha}{\gamma u}.$$
We see that the angle β is determined by two competing terms: γu and $g\cos\alpha$, as one would expect.
2. On the x direction,
$$dp_x = -mg\sin\alpha = m\dot{v}_x - \gamma m u \cos\beta \quad \rightarrow \quad v_x = (\gamma u \cos\beta - g\sin\alpha)\, t,$$
where the constant from the integration is zero, due to the initial condition $|\boldsymbol{v}(t=0)| = 0$.