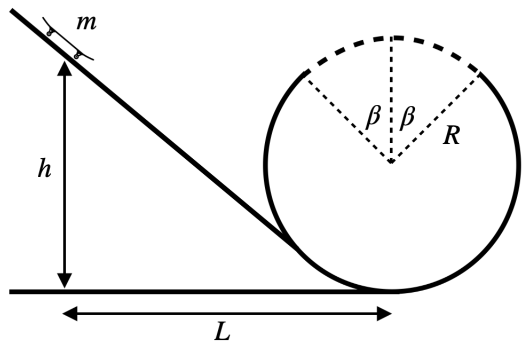
HW 7

1 Loop The Loop

A skate with mass m is sliding on a frictionless track which begins with a slope and ends with a loop with radius R, as shown in the figure.



1. What is the minimal height h, for which the skate will not leave the track at any point?
2. Due to faulty construction, the top part of the track (corresponds to angle 2β) collapsed. What should be the value of h so that the skate would complete a full loop on the track? Which value of β corresponds to minimal height h?
3. Consider the section of motion between $0 < x < L$ (where L is the distance to the bottom of the loop) and show that the velocity of the skate in the x direction follows

$$\dot{x} = \sqrt{\frac{2g\left(h - y\left(x\right)\right)}{1 + \left(\frac{dy}{dx}\right)^2}}.$$

Solution:

1. For the skate to stay on the track we require that the normal force between the skate and the track will be $N \geq 0$ at all times, where equality denotes the limit in which the skate parts from the track. Thus we need to look into the limit scenario in order to find the minimal height. Since the force which acts to pull the skate from the track is the gravitational force, the point in which its y component is largest is at the top of the loop.
Writing the equation of motion for the skate, at the top of the loop we find

$$N - mg = ma_r \quad \overset{N=0}{\longrightarrow} \quad v^2 = gR.$$

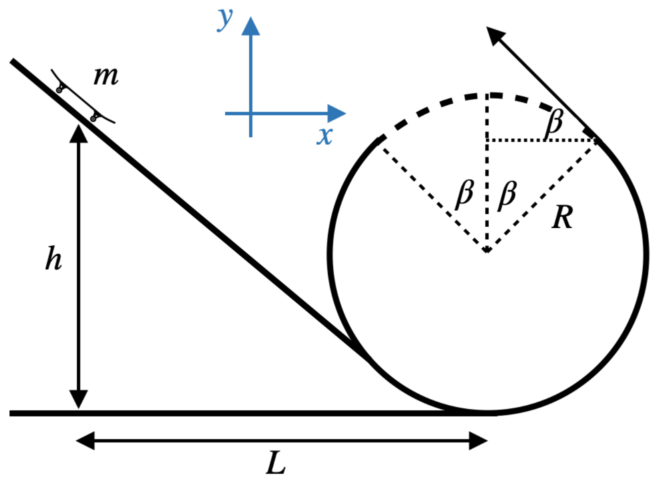
In order to find the relation between h and v^2 we employ conservation of energy

$$mgh = mg\left(2R\right) + \frac{1}{2}mv^2 \quad \to \quad h = \frac{v^2}{2g} + 2R,$$

plugging v^2 yields

$$h = \frac{5R}{2}.$$

2. Throughout the motion along the collapsed part of the track the skate undergoes free-fall motion. Therefore, we need to require that the skate's displacement in the horizontal direction is $2R\sin\beta$ as it reaches the same height it was when it left the track.



The initial velocity of the skate when it leaves the track is found from energy conservation

$$mgh = mgR\left(1+\cos\beta\right) + \frac{1}{2}mv_0^2 \quad \to \quad v_0^2 = 2g\left[h - R\left(1+\cos\beta\right)\right].$$

The time it takes for the skate to return to the same height is found by considering kinematics vertical direction:

$$0 = v_0\sin\beta\, t - \frac{g}{2}t^2 \quad \to \quad t_f = 2\frac{v_0\sin\beta}{g},$$

thus, the horizontal displacement is

$$|x| = v_0\cos\beta\left(2\frac{v_0\sin\beta}{g}\right) = 2\frac{v_0^2}{g}\cos\beta\sin\beta \overset{!}{=} 2R\sin\beta \quad \to \quad \frac{v_0^2}{g}\cos\beta = R.$$

Plugging in the expression for v_0^2 we find

$$2\left[h - R\left(1+\cos\beta\right)\right]\cos\beta = R \quad \to \quad h = R\left(\frac{1}{2\cos\beta} + 1 + \cos\beta\right).$$

In order to find the minimal value of h we take the derivative and find β which minimizes h

$$\frac{dh}{d\beta} = R\left(\frac{\sin\beta}{2\cos^2\theta} - \sin\beta\right) = 0 \quad \to \quad \cos\beta = \frac{1}{\sqrt{2}} \quad \to \quad \beta = \frac{\pi}{4}.$$

This result is reasonable since we know that 45° throw yields maximum distance for certain velocity, which can be reversed to minimal velocity for certain distance.

3. Let us write down the conservation of energy for any point on the slope

$$mgh = mgy + \frac{1}{2}mv^2,$$

but $v^2 = \dot{x}^2 + \dot{y}^2$, thus

$$\dot{x}^2 + \dot{y}^2 = 2g\left(h - y\right).$$

Since we need to find $\dot{x}\left(x\right)$ let us convert the time derivative of y to

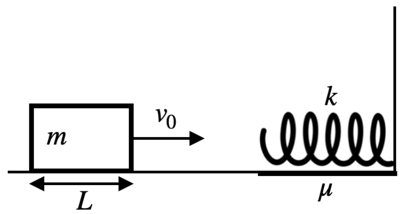
$$\dot{y} = \frac{dy}{dx}\dot{x},$$

hence

$$\dot{x} = \sqrt{\frac{2g\left(h - y\left(x\right)\right)}{1 + \left(\frac{dy}{dx}\right)^2}}.$$

2 Mass, Spring and Friction

A body with length L, with a uniform mass distribution and mass m is moving on a frictionless surface with velocity v_0. The body then reaches a region of the surface with friction coefficient $\mu = \mu_k = \mu_s$, on which lies a massless spring with coefficient k, which is connected to a wall, as shown in the figure.



1. What is the value of v_0, for which the body will stop exactly when its entire length is on the surface with the friction?
2. Given v_0 is the value from (1), what is the maximal value of μ that allows the complete return of the body?
3. Given v_0 is the value from (1) and μ from (2), what would be the body's final velocity?

Solution:

1. The equation for conservation of energy yields

$$\underbrace{\frac{1}{2}mv_0^2}_{E_k} = \underbrace{\frac{1}{2}kL^2}_{U_e} + \underbrace{|W_f|}_{\text{energy loss}} .$$

The work done by the friction force is

$$W_f = \int_0^L \mu N dx,$$

where the normal force is x dependent $N(x) = \frac{x}{L}mg$. Thus

$$W_f = \mu \frac{mg}{L} \int_0^L x\,dx = \frac{\mu mgL}{2} \quad \to \quad v_0 = \sqrt{\frac{k}{m}L^2 + \mu gL}.$$

2. Writing the expressions for both forces: elastic and friction for the interval $0 < x < L$, we find that

$$F_e = -kx$$
$$f = \mu \frac{x}{L} mg.$$

In order to have movement to the left we must require

$$F_e \geq f \quad \to \quad \mu \geq \frac{Lk}{mg}.$$

3. Again, using energy conservation

$$\frac{1}{2}kL^2 = \frac{1}{2}mv_f^2 + |W_f| \quad \to \quad v_f = \sqrt{\frac{k}{m}L^2 - \mu gL}.$$

3 The Worm Problem

A worm of mass m and of length L rest on a horizontal table with 1 quarter of it hanging over the edge of the table.
Between the table and the worm there is a friction constant μ.
At $t = 0$ the worm is free to slip from the table.
Solve using energy considerations:

1. What will be the velocity of the worm after its entire length is detached from the table?
2. For which range of values of μ the worm will slip and not remain static?

Solution:

1. The difference in kinetic energy is
$$\Delta k = \frac{1}{2} m v_f^2$$
where v_f is the velocity of the worm when it off the table.
Three forces act on the worm: gravity (mg), friction (μN), and the normal (N).
The normal force is always perpendicular to the motion and therefore has no work.
Let us assume that the worm mass is uniform and define a longitudinal mass density $\lambda = \frac{m}{L}$.
We also define l - the length of the worm which is hanging.
The friction is working on the part which is on the table and moving horizontally
$$W_{friction} = \int_{\frac{L}{4}}^{L} \left(-\mu\lambda g(L-l)\right)\hat{x} \cdot dl\hat{x} = -\left[\mu\lambda g\left(Ll - \frac{l^2}{2}\right)\right]_{\frac{L}{4}}^{L} = -\mu\lambda g\frac{9L^2}{32} = -\frac{9}{32}\mu mgL$$
since $N = \lambda g(L-l)$.
The gravitational force is working on the part falling from the table
$$W_g = \int_{\frac{L}{4}}^{L} \left(-\lambda gl)\right)\hat{y} \cdot dl\left(-\hat{y}\right) = \lambda g\left[\frac{l^2}{2}\right]_{\frac{L}{4}}^{L} = \frac{15}{32} mgL$$
So summing both force's work and using the work - energy theorem
$$\frac{1}{2} m v_f^2 = \frac{3}{32} mgL\left(5 - 3\mu\right)$$
$$v_f = \sqrt{\frac{3}{16} gL\left(5 - 3\mu\right)}.$$
2. The worm will start to slip only if $v_f > 0$
$$5 - 3\mu > 0$$
$$\mu < \frac{5}{3}.$$

4 Work of a changing force

A changing force acting on a mass of $5kg$ with an initial velocity of $4\frac{m}{sec}$.
The force depends on the displacement x and is different for different values of x according to:

$$\begin{array}{ll} F_1(x) = 2x & 0 < x < 5 \\ F_2(x) = 10 & 5 < x < 15 \\ F_3(x) = 20 - 2x & 15 < x < 20 \end{array}$$

a. What is the work of this force along each different section of x?

b. What will be the change in kinetic energy after the mass passed $20\,meters$? Assume this is the only force on the mass.

c. What is the velocity of the mass after $20\,meters$?

Solution:

a.

$$W_1 = \int_0^5 2x\,dx = \left[x^2\right]_0^5 = 25\,joule$$

$$W_2 = \int_5^{15} 10\,dx = [10x]_5^{15} = 100\,joule$$

$$W_3 = \int_{15}^{20} 40 - 2x\,dx = \left[40x - x^2\right]_{15}^{20} = 25\,joule$$

b. According to work - energy theorem the change in kinetic energy is given by

$$\Delta K = W_{tot} = W_1 + W_2 + W_3 = 150\,joule$$

c.

$$\Delta K = \frac{1}{2}m\left(v_f^2 - v_0^2\right)$$

$$v_f^2 = v_0^2 + \frac{2\Delta K}{m} = 16\frac{m^2}{sec^2} + \frac{2 \cdot 150}{5}\frac{joule}{kg}$$

Note that $\frac{joule}{kg} = \frac{m^2}{sec^2}$, and we get

$$v_f = 8.7\frac{m}{sec}.$$

5 Work on a box

A box with a mass of $m = 150\,kg$ hangs from a rope of length $L = 15\,meters$.
A varying force $\boldsymbol{F}$push the box horizontally to move it a distance of $d = 4\,meters$ so that at the end of the movement the mass remains motionless as shown in the figure.

A. What is the magnitude of the force $\boldsymbol{F}$ at the end of the motion?

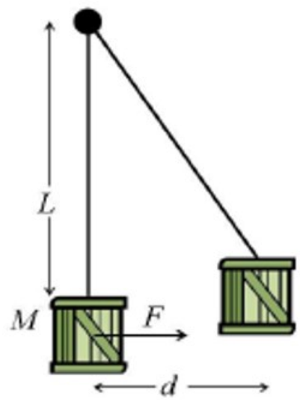
B. What is the total work done on the box during its movement?

C. What work is done on the box during its movement by gravity?

D. What is the work done on the box during its movement by the rope?

E. Given that before and after the motion the box remains in place. Use the answers to the previous sections to find the work done by the varying force $\boldsymbol{F}$.

F. Why is the work of $\boldsymbol{F}$ not equal to the product of the displacement in your answer to section **A**?



Solution:

A. The equation of motion when the box is it rest are:

$$\begin{array}{ll} \hat{x} & F - T\sin\theta = 0 \\ \hat{y} & T\cos\theta - mg = 0 \end{array}$$

The angle between the rope at the end and the vertical direction is given by

$$\sin\theta = \frac{d}{L} = \frac{4}{15}.$$

Then $T = \frac{mg}{\cos\theta}$ and

$$F = mg\tan\theta = 415\,N$$

B. The kinetic energy at the start and at the end of the motion is zero.
According to work - energy theorem

$$W_{tot} = \Delta K = 0$$

C. The location of the box is given by (origin at the upper end of the rope)

$$\boldsymbol{r} = (L\sin\theta, -L\cos\theta)$$

then the trajectory is given by

$$\boldsymbol{dl} = L\,(\cos\theta, \sin\theta)\,d\theta$$

for $\theta : 0 \to \theta$.

$$W_g = \int_0^\theta -mg\hat{y}\cdot L\,(\cos\theta,\sin\theta)\,d\theta = \int_0^\theta -mgL\sin\theta\,d\theta =$$

$$= [mgL\cos\theta]_0^\theta = mgL\left(\frac{L}{\sqrt{L^2+d^2}} - 1\right) = -815\,joule$$

D. The tension force acting on the mass by the rope is perpendicular to the trajectory during the whole motion.

$$W_{rope} = 0$$

E.

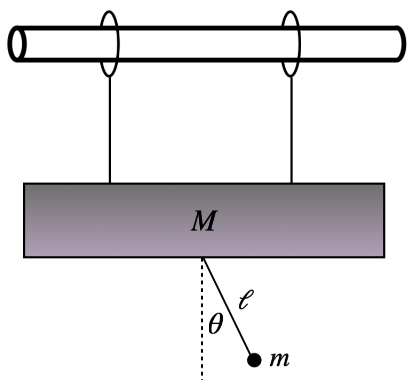
$$0 = W_{tot} = W_F + W_g + \cancel{W_{rope}}$$
$$W_F = -W_g = 815\, joule$$

F.

$$F(\theta) \neq const$$
$$W_F = \int F dl \neq F \int dl.$$

6 Sliding Pendulum - Bonus

A body with mass M is hanged by from two massless loops on a frictionless cylinder. A massless rod with length ℓ connects between the center of the body and a ball with mass m, as shown in the figure. The ball is released from an unknown angle and the system begins to move.



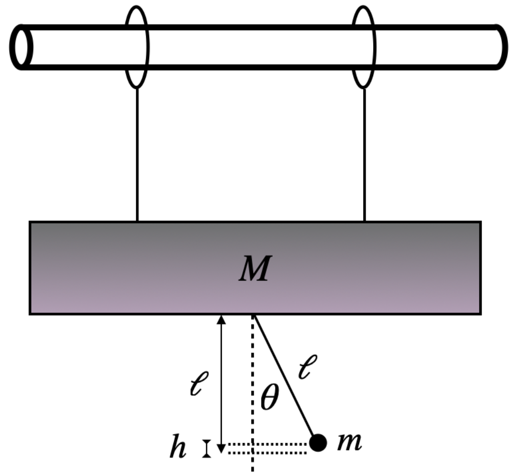
1. Find the ratio between the horizontal velocity of mass M and that of mass m. (*Hint*: treat both masses as one system)
2. Find an expression for the kinetic and potential energy of the system, as a function of the angle θ and its time derivative $\dot{\theta}$ (and the other given quantities of the system: m, M and ℓ). (*Hint*: use the ratio you found)
3. Use energy conservation to show that for small angles the system follows the equation: $\ddot{\theta} + \omega^2\theta = 0$, and find ω.

Solution:

1. There are no external forces on the horizontal direction, therefore momentum is conserved in this direction

$$p_i = p_f \quad \rightarrow \quad 0 = MV_x + mv_x \quad \rightarrow \quad \frac{V_x}{v_x} = -\frac{m}{M}.$$

2. The potential energy is only due to the mass m (since M remains in the same height).



Expressing the potential energy in terms of the angle θ reads

$$E_p = mg\ell(1-\cos\theta).$$

In order to find the total kinetic energy we start from the motion of m relative to M, which is a circular movement with constant radius $r=\ell$. Therefore, the relative velocity is $\ell\dot{\theta}\left(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{y}}\right)$, and the velocity relative to stationary observer is

$$\boldsymbol{v} = V_x\hat{\boldsymbol{x}} + \ell\dot{\theta}\left(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{y}}\right).$$

But we already know the ratio V_x/v_x, thus

$$v_x = \ell\dot{\theta}\cos\theta + V_x = \ell\dot{\theta}\cos\theta - \frac{m}{M}v_x \quad \rightarrow \quad v_x = \frac{M}{m+M}\ell\dot{\theta}\cos\theta$$

$$V = V_x = -\frac{m}{m+M}\ell\dot{\theta}\cos\theta.$$

The kinetic energy is therefore

$$\begin{aligned} E_k &= \frac{1}{2}m\left(v_x^2+v_y^2\right) + \frac{1}{2}MV^2 \\ &= \frac{1}{2}m\left(\frac{M^2}{(m+M)^2}\ell^2\dot{\theta}^2\cos^2\theta + \ell^2\dot{\theta}^2\sin^2\theta\right) + \frac{1}{2}M\frac{m^2}{(m+M)^2}\ell^2\dot{\theta}^2\cos^2\theta \\ &= \frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}m\ell^2\dot{\theta}^2\sin^2\theta, \end{aligned}$$

and the total energy is

$$E = mg\ell(1-\cos\theta) + \frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}m\ell^2\dot{\theta}^2\sin^2\theta.$$

3. For small angles

$$\sin\theta \simeq \theta \quad \text{and} \quad \cos\theta \simeq 1-\frac{1}{2}\theta^2,$$

therefore the total energy is

$$\begin{aligned} E &\simeq \frac{1}{2}mg\ell\theta^2 + \frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2\left(1-\frac{1}{2}\theta^2\right)^2 + \frac{1}{2}m\ell^2\dot{\theta}^2\theta^2 \\ &\simeq \frac{1}{2}mg\ell\theta^2 + \frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2, \end{aligned}$$

which is conserved:

$$\dot{E} = mg\ell\theta\dot{\theta} + \frac{mM}{(m+M)}\ell^2\dot{\theta}\ddot{\theta} = 0 \quad \rightarrow \quad \ddot{\theta} + \frac{m+M}{M}\frac{g}{\ell}\dot{\theta} = 0,$$

thus

$$\omega = \sqrt{\frac{m+M}{M}\frac{g}{\ell}}.$$