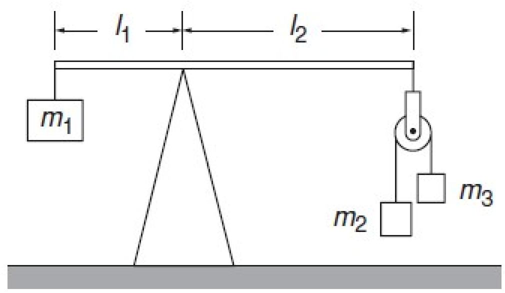
HW 9

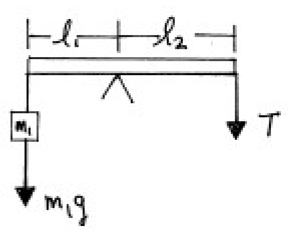
1 Beam and Atwood's machine

A pivoted beam has a mass m_1 suspended from one end and an Atwood's machine suspended from the other (see left-hand sketch).
The friction less pulley has negligible mass and dimension.
Gravity is directed downward, and $m_2 > m_3$.
Find a relation between m_1,m_2,m_3,l_1, and l_2 that will ensure that the beam has no tendency to rotate just after the masses are released.



Solution:

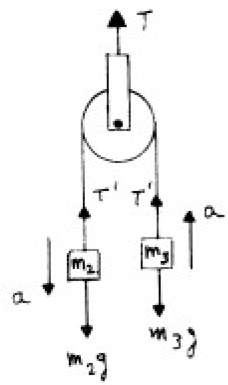
The beam is in equilibrium, so the total torque= 0.
Taking torques about the rotating axis,



$$Tl_2 - m_1gl_1 = 0$$

$$T = m_1g\frac{l_1}{l_2}$$

The equations of motion for m_2 and $m_3$are



$$m_2 a = m_2 g - T' \quad m_3 a = T' - m_3 g$$

$$T' = 2g\left(\frac{m_2 m_3}{m_2 + m_3}\right)$$

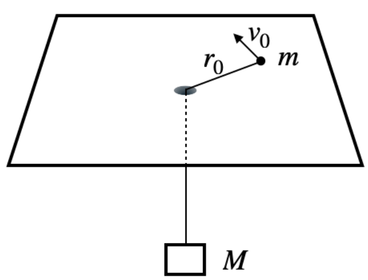
In equilibrium, the pulley does not accelerate.

$$T = 2T' = 4g\left(\frac{m_2 m_3}{m_2 + m_3}\right)$$

$$m_1 g \frac{l_1}{l_2} = 4g\left(\frac{m_2 m_3}{m_2 + m_3}\right)$$

2 Masses On a Table

We revisit the system of two masses and a table: A mass m, placed on a frictionless table, is connected to another mass M via string, which goes through a hole in the center of the table. at $t = 0$ the mass m is at distance r_0 from the hole and moves with velocity v_0 counterclockwise.



1. Find the equation of motion for $r(t)$ using energy and angular momentum conservation laws (*Hint*: obtain a second order differential equation, no need to solve it).
2. For which r_0 will m continue to move with constant radius?

Solution:

1. Let us define the length of the string to be L and the potential energy point of reference at the table's height. Energy conservation states that

$$-Mg\left(\not{L} - r\right) + \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(r\dot{\theta}\right)^2 = Mg\left(\not{L} - r_0\right) + \frac{1}{2}mv_0^2,$$

where we used the relation $z = L - r \to \dot{z} = -\dot{r}$, we also separated the kinetic energy of m into radial and tangent components $\left(\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right)$. Since all the forces on m are perpendicular to the radial direction we conclude that angular momentum is also conserved, hence

$$\begin{aligned}\boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} \\ &= r\hat{\boldsymbol{r}} \times m\left(\dot{r}\not{\hat{\boldsymbol{r}}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) \\ &= mr^2\dot{\theta}\hat{\boldsymbol{z}} = mrv\hat{\boldsymbol{z}},\end{aligned}$$

which gives us

$$mr^2\dot{\theta} = mr_0v_0.$$

Plugging $\dot{\theta}$ into the energy equation yields

$$Mgr + \frac{1}{2}\left(M+m\right)\dot{r}^2 + \frac{1}{2}m\left(\frac{r_0}{r}v_0\right)^2 = -Mgr_0 + \frac{1}{2}mv_0^2,$$

taking the time derivative would give us

$$Mg\dot{r} + \left(M+m\right)\dot{r}\ddot{r} - m\frac{r_0^2v_0^2}{r^3}\dot{r} = 0$$

$$\ddot{r} + \frac{Mg}{M+m} - \frac{m}{M+m}\frac{r_0^2v_0^2}{r^3} = 0.$$

2. For motion with constant radius r_0 we require

$$a_r = -\frac{v_0^2}{r_0},$$

plugging into the force equations

$$\begin{aligned}T &= -m\frac{v_0^2}{r_0} \\ T - Mg &= 0\end{aligned}$$

gives us

$$r_0 = \frac{m}{M}\frac{v_0^2}{g}.$$

3 Angular Momentum

Calculate the angular momentum and torque for the following cases, once for rotation around the origin and once for rotation around $(0, -R)$:

1. A particle with mass m, moves on a circular trajectory with radius R counterclockwise, around the origin, with angular velocity ω. Assume that $\boldsymbol{r}\left(t=0\right) = -R\hat{\boldsymbol{y}}$.

2. A particle with mass m, moves along the x axis with $y = -R$, and velocity v. Assume that $\boldsymbol{r}\left(t=0\right) = -R\hat{\boldsymbol{y}}$.

Solution:

1. Around the origin:
   The angular momentum is

$$\begin{aligned}\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} &= R\left[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}\right] \times mR\omega\left[-\sin(\omega t)\,\hat{\boldsymbol{x}} + \cos(\omega t)\,\hat{\boldsymbol{y}}\right] \\ &= mR^2\omega\left[\cos^2(\omega t) + \sin^2(\omega t)\right]\hat{\boldsymbol{z}} \\ &= mR^2\omega\hat{\boldsymbol{z}}.\end{aligned}$$

   The torque is

$$\begin{aligned}\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} &= \boldsymbol{r} \times m\boldsymbol{a} \\ &= \boldsymbol{r} \times m\left(-R\omega^2\hat{\boldsymbol{r}}\right) \\ &= 0.\end{aligned}$$

   Around $-R\hat{\boldsymbol{y}}$:
   The angular momentum is

$$\begin{aligned}\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} &= R\left[\cos(\omega t)\,\hat{\boldsymbol{x}} + (1 + \sin(\omega t))\,\hat{\boldsymbol{y}}\right] \times mR\omega\left[-\sin(\omega t)\,\hat{\boldsymbol{x}} + \cos(\omega t)\,\hat{\boldsymbol{y}}\right] \\ &= mR^2\omega\left[\sin(\omega t) + \cos^2(\omega t) + \sin^2(\omega t)\right]\hat{\boldsymbol{z}} \\ &= mR^2\omega\left[1 + \sin(\omega t)\right]\hat{\boldsymbol{z}}.\end{aligned}$$

   The torque is

$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = mR^2\omega^2\cos(\omega t)\,\hat{\boldsymbol{z}}.$$

2. Around the origin:
   The angular momentum is

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = (vt\hat{\boldsymbol{x}} - R\hat{\boldsymbol{y}}) \times mv\hat{\boldsymbol{x}} = Rmv\hat{\boldsymbol{z}}.$$

   The torque is

$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = 0.$$

   Around $-R\hat{\boldsymbol{y}}$:
   The angular momentum is

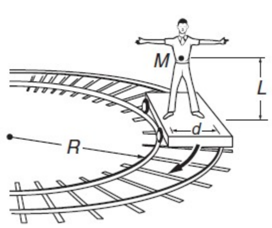
$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = (vt\hat{\boldsymbol{x}}) \times mv\hat{\boldsymbol{x}} = 0.$$

   The torque is

$$\boldsymbol{\tau} = 0.$$

4 Man on a railroad car

A man of mass M stands on a railroad car that is rounding an unbanked turn of radius R at speed v.
His center of mass is height L above the car, and his feet are distance d apart.
The man is facing the direction of motion. How much weight is on each of his feet?



Solution:

Denote N_1 for the man's outer foot and N_2 for the inner foot.
Vertical equation of motion

$$N_1 + N_2 - Mg = 0$$

Radial equation of motion

$$f_1 + f_2 = M\frac{v^2}{R}$$

Torque about the center of mass

$$-N_1\frac{d}{2} + N_2\frac{d}{2} + (f_1 + f_2)\,L = 0$$

$$\frac{d}{2}(N_1 - N_2) = mv^2\frac{L}{R}$$

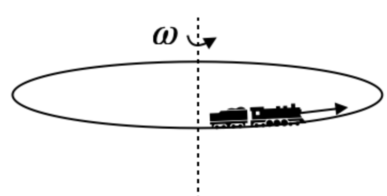
And we get

$$N_1 = \frac{1}{2}\left(Mg + \frac{Mv^2}{R}\frac{2L}{d}\right)$$

$$N_2 = \frac{1}{2}\left(Mg - \frac{Mv^2}{R}\frac{2L}{d}\right)$$

5 A Toy Train

A toy train is mounted on a large horizontal wheel (mass M, radius R) that can turn about a vertical axis.
The train has mass m, and with the system initially at rest, its electric motor is turned on.
The train reaches a steady speed v with respect to the track.
What is the angular velocity ω of the wheel?



Solution:

The system is isolated hence angular momentum is conserved.
In the beginning the system is at rest

$$L_i = 0$$

After the train got a steady speed

$$L_f = Rm\,(v + \omega R) + MR^2\omega$$

Using angular momentum conservation

$$L_f = L_i$$

$$-Rmv = (M + m)\,R^2\omega$$

$$\omega = -\frac{mv}{(M + m)\,R}$$

The minus sign tells us that the train and the wheel must turn in opposite directions for the angular momentum to be conserved.

6 Satellite

A satellite with mass m moves in space under gravitational force of a planet $\boldsymbol{F} = -\frac{k}{r^2}\hat{\boldsymbol{r}}$, where k is a known constant, r is the distance of the satellite from the center of the planet and $\hat{\boldsymbol{r}}$ is a unit vector directed from the center of the planet to the satellite. In addition, the satellite experiences a drag force $\boldsymbol{F}_d = -\beta\boldsymbol{v}$ in the opposite direction to its velocity, where β is a known constant.

1. Write down the equations of motion of the satellite. *Notice*:
   (a) Its motion is not necessarily circular.
   (b) Since the motion is on a plain, a set of 2 equations is required to describe the motion completely.\
   (c) Any choice of coordinate system is valid.
   (d) There is no need for solving the equations of motion.
2. Assuming that at $t = 0$ the satellite was at distance R_0 from the planet's center, and its angular velocity was ω_0, find the angular momentum of the satellite with respect to the center of the planet as a function of time.

Solution:

1. Using polar coordinates, the velocity of the satellite is

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\varphi}\hat{\boldsymbol{\varphi}},$$

thus

$$\boldsymbol{F}_d = -\beta\dot{r}\hat{\boldsymbol{r}} - \beta r\dot{\varphi}\hat{\boldsymbol{\varphi}}.$$

Plugging the drag force into the force equation for the radial axis, along with the gravitational force, we find

$$-\beta\dot{r} - \frac{k}{r^2} = ma_r \quad \rightarrow \quad -\frac{\beta}{m}\dot{r} - \frac{k}{m}\frac{1}{r^2} = \ddot{r} - r\dot{\varphi}^2.$$

Whereas the equation for the tangent axis reads

$$-\beta r\dot{\varphi} = ma_t \quad \rightarrow \quad -\frac{\beta}{m}r\dot{\varphi} = 2\dot{r}\dot{\varphi} + r\ddot{\varphi}.$$

2. The initial angular momentum is

$$\boldsymbol{L}\,(t=0) = \boldsymbol{r}\,(t=0) \times m\dot{\boldsymbol{r}}\,(t=0) = mR_0^2\omega_0\hat{\boldsymbol{z}}$$

while for any time t

$$\boldsymbol{L} = \boldsymbol{r} \times m\dot{\boldsymbol{r}} = r\hat{\boldsymbol{r}} \times m\left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\varphi}\hat{\boldsymbol{\varphi}}\right) = mr^2\dot{\varphi}\hat{\boldsymbol{z}}.$$

The torque is

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = r\hat{\boldsymbol{r}} \times \left[-\left(\beta\dot{r} + \frac{k}{r^2}\right)\hat{\boldsymbol{r}} - \beta r\dot{\varphi}\hat{\boldsymbol{\varphi}}\right] = -\beta r^2\dot{\varphi}\hat{\boldsymbol{z}} = -\frac{\beta}{m}\boldsymbol{L}.$$

Using the relation $\boldsymbol{\tau} = \dot{\boldsymbol{L}}$we find

$$\dot{\boldsymbol{L}} = -\frac{\beta}{m}\boldsymbol{L},$$

which is a fist order differential equation in the z axis (there is no angular momentum in the other directions), solved by

$$\frac{dL}{L} = -\frac{\beta}{m}dt \quad \rightarrow \quad L = Ce^{-\frac{\beta}{m}t},$$

where C is found from the initial conditions $L\,(t=0) = mR_0^2\omega_0$,

$$\boldsymbol{L}\,(t) = mR_0^2\omega_0 e^{-\frac{\beta}{m}t}\hat{\boldsymbol{z}}.$$

7 Collision with an Angle

Two small masses connected by a mass less rod of length $2l$.
A third mass hit elastically one of the connected masses with a velocity of v_0 in a direction vertical to the rod.
After the collision, the angle between the rod COM velocity and the initial velocity direction is α.
The three masses are identical.
Find the angular velocity ω of the rod and two masses around their center of mass after the collision.

Solution:

In this collision the energy (elastic collision), angular momentum, and linear momentum all conserved.
For a body that consists of smaller parts moving all together the linear momentum means the momentum of the center of mass.
In the initial state, let us choose the rod to be aligned with the y axis where the bottom mass located on the origin.
The third mass was chosen to move along the x axis with $\boldsymbol{v} = v_0\hat{\boldsymbol{x}}$.
Denote $\boldsymbol{u}$ for the velocity of the third mass after the collision.

- Linear momentum conservation

$$m v_0 \hat{\boldsymbol{x}} = m\boldsymbol{u} + 2m\boldsymbol{v_{cm}}$$
$$u_y = -2v_{cm,y} \Rightarrow v_{cm,y} = -\frac{u_y}{2}$$
$$u_x = v_0 - 2v_{cm,x} \Rightarrow v_{cm,x} = \frac{v_0 - u_x}{2}$$

- Angular momentum conservation

$$m v_0 l \hat{\boldsymbol{z}} = m\boldsymbol{u} \times (-l\hat{\boldsymbol{y}}) + 2ml^2\omega\hat{\boldsymbol{z}}$$

Looking at the $\hat{z}$ component and divide by ml:

$$v_0 = u_x + 2l\omega \Rightarrow u_x = v_0 - 2l\omega$$

- Given angle α

$$\frac{v_{cm,y}}{v_{cm,x}} = \tan\alpha = \frac{u_y}{u_x - v_0}$$

Substituting u_x

$$\tan\alpha\,(v_0 - 2l\omega - v_0) = u_y$$
$$u_y = -2l\omega\tan\alpha$$
$$\boldsymbol{u} = (v_0 - 2l\omega)\,\hat{\boldsymbol{x}} - 2l\omega\tan\alpha\hat{\boldsymbol{y}}$$

and we can also find v_{cm}:

$$\boldsymbol{v_{cm}} = \frac{v_0 - (v_0 - 2l\omega)}{2}\hat{\boldsymbol{x}} - \frac{-2l\omega\tan\alpha}{2}\hat{\boldsymbol{y}}$$
$$\boldsymbol{v_{cm}} = l\omega\,(1\hat{\boldsymbol{x}} + \tan\alpha\hat{\boldsymbol{y}})$$

Now all the velocities are in terms of ω, l, v_0, and $\alpha$and we need one last equation to find ω.

- Energy conservation

$$m\frac{v_0^2}{2} = m\frac{u^2}{2} + 2m\frac{v_{cm}^2}{2} + 2m\frac{(l\omega)^2}{2}$$

Divide by $\frac{m}{2}$ and substituting $\boldsymbol{v_{cm}}$, and $\boldsymbol{u}$:

$$v_0^2 = (v_0 - 2l\omega)^2 + 4\,(l\omega)^2\tan^2\alpha + 2\,(l\omega)^2\left[1 + \tan^2\alpha\right] + 2\,(l\omega)^2$$
$$4l\omega v_0 = 2\,(l\omega)^2\left[4 + 3\tan^2\alpha\right]$$
$$\omega = \frac{v_0}{l}\frac{2}{4 + 3\tan^2\alpha}$$