HW 11

1 Motorcyclist

A motorcyclist decides to drive in an angular motion at a constant frequency ω, with its distance from the center changing as $r = bt$, where b is constant.
How far from the center of motion can she reach if the coefficient of friction is μ?
You may assume $\mu g \gg \omega b$.

Solution:

In a polar coordinate system

$$\boldsymbol{r} = r\hat{r}$$

$$\boldsymbol{v} = \dot{r}\hat{r} + r\omega\hat{\varphi}$$

$$\hat{v} = \frac{\dot{r}\hat{r} + r\omega\hat{\varphi}}{\sqrt{\dot{r}^2 + \omega^2 r^2}}$$

$$\boldsymbol{a} = \left(\ddot{r} - \omega^2 r\right)\hat{r} - 2\dot{r}\omega\hat{\varphi}$$

Let's write equations of motion using Newton's second law

$$\Sigma \boldsymbol{F} = -\mu N\hat{v} + N\hat{z} - mg\hat{z} = m\boldsymbol{a}$$

There is no motion in the vertical direction

$$N = mg$$

and we left with

$$-\mu mg \frac{\dot{r}\hat{r} + r\omega\hat{\varphi}}{\sqrt{\dot{r}^2 + \omega^2 r^2}} = m\left(\ddot{r} - \omega^2 r\right)\hat{r} - 2m\dot{r}\omega\hat{\varphi}$$

In the $\hat{\varphi}$ direction

$$-\mu \cancel{m} g \frac{r\cancel{\omega}}{\sqrt{\dot{r}^2 + \omega^2 r^2}} = -2\cancel{m}\dot{r}\cancel{\omega}$$

Taking a power of 2

$$\mu^2 g^2 r^2 = 4\dot{r}^2\left(\dot{r}^2 + \omega^2 r^2\right)$$

$$r^2\left(\mu^2 g^2 - 4\omega^2\dot{r}^2\right) = 4\dot{r}^4$$

$$r = \frac{2\dot{r}^2}{\sqrt{\mu^2 g^2 - 4\omega^2\dot{r}^2}} = \frac{2\dot{r}^2}{\mu g\sqrt{1 - 4\left(\frac{\omega\dot{r}}{\mu g}\right)^2}}$$

where $\dot{r} < b$ and then

$$\frac{\omega\dot{r}}{\mu g} < \frac{\omega b}{\mu g} \ll 1$$

therefotre this term can be neglected and we have

$$r = \frac{2\dot{r}^2}{\mu g}$$

In the radial direction

$$-\mu \not{m} g \frac{\dot{r}}{\sqrt{\dot{r}^2 + \omega^2 r^2}} = \not{m} \left(\ddot{r} - \omega^2 r\right)$$

Replacing r

$$-\frac{\mu g}{\sqrt{1 + \cancel{\frac{4\omega^2 \dot{r}^2}{\mu^2 g^2}}}} = \ddot{r} - \omega^2 \frac{2\dot{r}^2}{\mu g}$$

Neglecting the same term for the same reasons as before-

$$-\mu g = \ddot{r} - \omega^2 \frac{2\dot{r}^2}{\mu g}$$

$$\frac{d}{dt}\dot{r} = -\mu g + 2\frac{\omega^2}{\mu g}\dot{r}^2$$

$$\frac{d\dot{r}}{\mu g - 2\frac{\omega^2}{\mu g}\dot{r}^2} = \frac{d\dot{r}}{\mu g}\frac{1}{1 - 2\cancel{\left(\frac{\omega \dot{r}}{\mu g}\right)^2}} = \frac{d\dot{r}}{\mu g} = -dt$$

Integrating by time from $t = 0$ to some later t with the initial conditions of $\dot{r}(t = 0) = b$

$$\dot{r}(t) = b - \mu g t$$

So the motorcyclist stops at

$$t = \frac{b}{\mu g}$$

And at that moment her distance from the center is

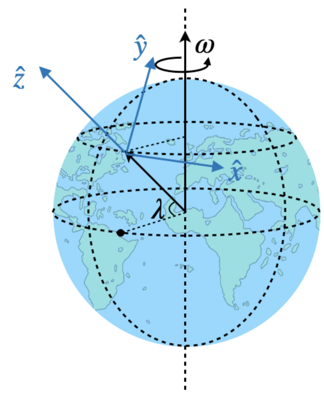
$$d = \int_0^{\frac{b}{\mu g}} (b - \mu g t)\, dt = \frac{b^2}{\mu g} - \frac{1}{2}\frac{b^2}{\mu g} = \frac{1}{2}\frac{b^2}{\mu g}$$

2 Physicist Pilot

A pilot, which is also a physicist, takes off the ground. The pilot, which knows all about the Coriolis force, wants to take it easy on the airplane. To which direction should he take off, east or west? Explain. (If you have any doubt - earth rotates eastwards).

Solution:

Let us define the coordinate system as follows: z in the radial direction of the earth, y points to the north, x points to the east:



In this coordinate system, the angular velocity vector is

$$\boldsymbol{\omega} = \omega\left(0, \cos\lambda, \sin\lambda\right).$$

Let us also denote the takeoff velocities

$$\begin{aligned} \boldsymbol{v}_E &= v\hat{\boldsymbol{x}}, \\ \boldsymbol{v}_W &= -v\hat{\boldsymbol{x}}. \end{aligned}$$

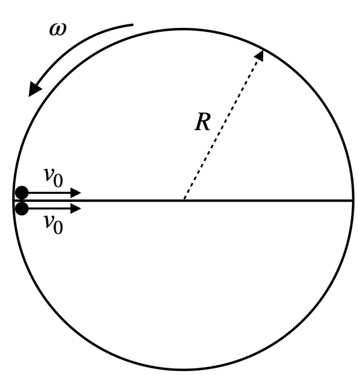
The Coriolis force is

$$-2m\boldsymbol{\omega}\times\boldsymbol{v} = -2m\omega v_i \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & \cos\lambda & \sin\lambda \\ 1 & 0 & 0 \end{vmatrix} = 2m\omega v_i \begin{pmatrix} 0 \\ -\sin\lambda \\ \cos\lambda \end{pmatrix},$$

where $v_i = \pm v$. We find that the if the plane takes off to the east, $v_i = v$, then the Coriolis force acts in the positive z direction, assisting the plane to gain altitude.

3 Two Masses on Disk

A horizontal frictionless disk with radius R is rotating at constant angular velocity ω. The disk also has a frictionless partition on it (as shown in the figure) and two point masses are positioned on the edge of the disk. The masses are moving with initial velocity v_0 (relative to the rotating disk) towards the center of the disk. What are the components of the acceleration for each mass at the beginning of the motion, relative to the disk? Which mass will move along the partition?



Solution:

Ignoring the gravitational force, which is only cancelled by the normal from the frictionless disk, we may write the II law of Newton in the frame of the disk as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{N} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega} + m\left(\boldsymbol{\omega} \times \boldsymbol{r}\right) \times \boldsymbol{\omega},$$

where the normal is due to contact with the partition and

$$\begin{aligned} \boldsymbol{\omega} &= \omega\hat{\boldsymbol{z}}, \\ \boldsymbol{r} &= R\hat{\boldsymbol{r}}, \\ \dot{\boldsymbol{r}} &= -v_0\hat{\boldsymbol{r}}, \\ \boldsymbol{N} &= N\hat{\boldsymbol{\varphi}}, \end{aligned}$$

the z direction points out of the page (towards us). Evaluating each force separately reads

$$\begin{aligned} 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega} &= 2m\omega v_0\hat{\boldsymbol{\varphi}}, \\ m\left(\boldsymbol{\omega} \times \boldsymbol{r}\right) \times \boldsymbol{\omega} &= m\omega^2 R\hat{\boldsymbol{r}}. \end{aligned}$$

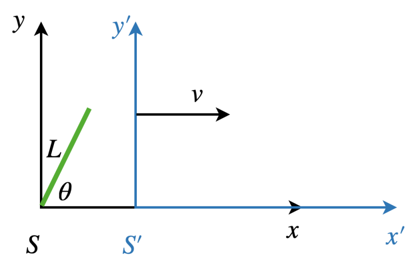
In order for a mass to stay in contact with the partition we require $\boldsymbol{a}_{\varphi} = 0$, thus

$$N + 2m\omega v_0 = 0 \quad \rightarrow \quad \boldsymbol{N} = -2m\omega v_0\hat{\boldsymbol{\varphi}},$$

we find that the normal force should point clockwise, which means that the only mass that could remain in contact with the partition is the mass in the mass above the partition in the figure. The mass which is below the partition in the figure will accelerate counterclockwise, parting with the partition.

4 Relativistic Angle

A rod with length L is positioned with angle θ relative to the x axis in frame S. What would be the angle from the x axis of S, observed from frame S', which moves at $\boldsymbol{v} = v\hat{\boldsymbol{x}}$? What would be the length of the rod in S'?



Solution:

In the S frame

$$\begin{aligned} \Delta x &= L\cos\theta \\ \Delta y &= L\sin\theta, \end{aligned}$$

thus, in the S' frame

$$\begin{aligned} \Delta x' &= \frac{L\cos\theta}{\gamma}, \\ \Delta y' &= L\sin\theta. \end{aligned}$$

Therefore

$$\tan\theta' = \frac{\Delta y'}{\Delta x'} = \gamma \tan\theta,$$

the angle is larger in the S' frame, which is reasonable since only the length at the x direction is shortened due to the motion in the x direction.
The length of the rod in the S' frame is

$$L' = \sqrt{\Delta x'^2 + \Delta y'^2} = L\sqrt{\frac{\cos^2\theta}{\gamma} + \sin^2\theta} < L.$$

5 Relativistic Teacher

The exam in physics started at 9:00 and the teacher went for a walk at a speed of 0.8 of the speed of light (a particularly fast teacher).
After an hour by his clock he sends the students a radio signal to finish the exam.
How long did the exam last for the students.

Solution:

For the teacher $\beta = 0.8$ and $\gamma = \frac{1}{\sqrt{1-0.8^2}} = \frac{5}{3}$.
Event A - teacher sends a radio signal.
In the frame of the teacher

$$t_A = 1\,hour \;\; x_A = 0.$$

In the frame of the students

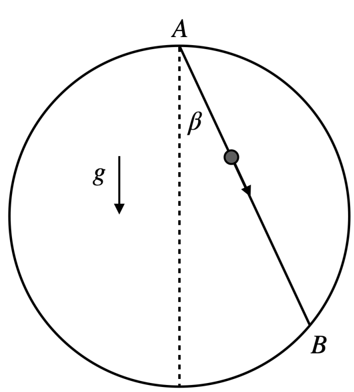
$$t'_A = \gamma\left(t_A - \beta\frac{x_A}{c}\right) = \frac{5}{3}\cdot 1\,hour$$

$$x'_A = \gamma\left(x_A - \beta c t_a\right) = -\frac{5}{3}\frac{4}{5}\cdot 1\,light\,hours$$

So from the students point of view the signal was sent at $10:40$ and it has a distance of $\frac{4}{3}\,light\,hours$ to travel (advancing with the speed of light - it will take $80\,minutes$ for the signal to reach the students).
For the students the exam ended at $12:00$ and last for $3\,hours$.

6 Sliding Bead

A string is connected from point A, at the top of a vertical circle, to point B in the circle. A bead is released from point A and slides along the frictionless string as shown in the figure.



1. Show that the time it takes the bead to reach from A to B is independent on the angle β between the string and the vertical axis.

2. The system is placed on an car, which accelerates to the right at value A. From which point A' should the bead be released, so that the time it take for it to slide along the string $A'B'$ will be independent of the string's length?

Solution:

1. Using geometry we find that, if the radius of the circle is R, then

$$AB = 2R\cos\beta.$$

Defining the coordinate system so that the x direction coincides with the string, we can write the force equation along the string as

$$mg\cos\beta = ma \quad \rightarrow \quad a = g\cos\beta,$$

which leads to

$$v = g\cos\beta\, t,$$

and

$$\ell = AB = \frac{1}{2}g\cos\beta\, t^2.$$

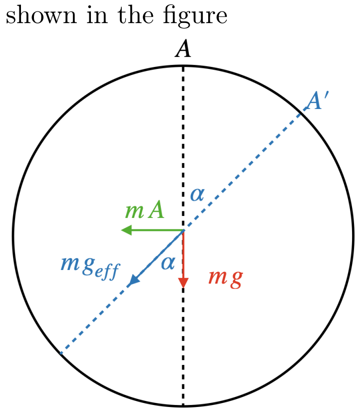
Plugging in the expression for AB we find

$$2R\cos\beta = \frac{1}{2}g\cos\beta\, t^2 \quad \rightarrow \quad t = 2\sqrt{R/g}.$$

2. First, let us note that the requirement that the time will not depend on the string's length is equivalent to the requirement that the time will not depend on β, since for any A', B' we will find $A'B' = 2R\cos\beta'$.
In the non-inertial frame of the circle

$$m\boldsymbol{a} = \boldsymbol{F} - m\boldsymbol{A}.$$

Let us denote the total acceleration as $\boldsymbol{g}_{\text{eff}}$. Now, our problem resembles the one from (1) only with the circle rotated by some angle α, as shown in the figure



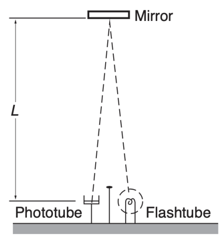
where

$$\tan\alpha = \frac{A}{g}.$$

7 Time Dilation

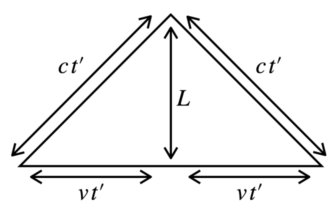
The clock in the sketch can provide an intuitive explanation of the time dilation formula. The clock consists of a flashtube, mirror, and phototube. The flashtube emits a pulse of light that travels distance L to the mirror and is reflected back to the phototube. Every time a pulse hits the phototube it triggers the flashtube. Neglecting time delay in the triggering circuits, the period of the clock is $\tau_0 = 2L/c$. Now examine the clock in a coordinate system moving to the left with uniform velocity v. In this system the clock appears to move to the right with velocity v. Find the period of the clock in the moving system by direct calculation, using only the assumptions that c is a universal constant, and that distance perpendicular to the line of motion is unaffected by the motion. The result should be identical to that given by the Lorentz transformation:

$$\tau = \tau_0/\sqrt{1 - v^2/c^2}.$$



Solution:

In the moving frame S' the motion of the light is described by the triangle:



Therefore

$$(ct')^2 = L^2 + (vt')^2 \quad \rightarrow \quad (t')^2 = \frac{L^2}{c^2 + v^2}.$$

The total time it takes for the light to complete the period is

$$\tau = 2t' = 2\frac{L}{c}/\sqrt{1 + v^2/c^2} = \tau_0/\sqrt{1 + v^2/c^2}.$$