Mathematical Background

1 Trigonometry

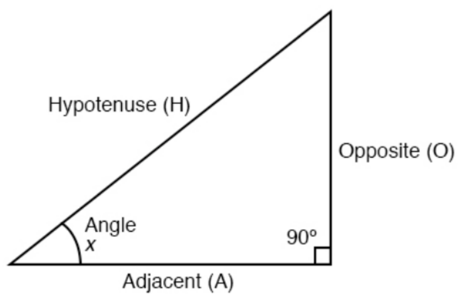


Figure 1: Right triangle illustration.

For a right triangle, illustrated in Fig. 1, we define the basic *trigonometric functions* sine, cosine and tangent by the length relations:

$$\begin{aligned} \sin x &= \frac{O}{H}, \\ \cos x &= \frac{A}{H}, \\ \tan x &= \frac{\sin x}{\cos x} = \frac{O}{A}. \end{aligned} \tag{1.1}$$

The inverse functions:

$$\begin{aligned} \arcsin \frac{O}{H} &= x, \\ \arccos \frac{A}{H} &= x, \\ \arctan \frac{O}{A} &= x. \end{aligned} \tag{1.2}$$

From *Pythagorean theorem* we know that

$$A^2 + O^2 = H^2, \tag{1.3}$$

then using (1.1) we get the identity

$$\cos^2 x + \sin^2 x = 1. \tag{1.4}$$

Additional useful trigonometric identities are:

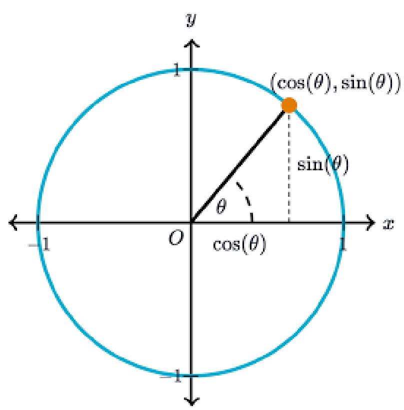
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \tag{1.5}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{1.6}$$

Notice that the mentioned above definitions are only valid for acute angle ($0^\circ \leq x \leq 90^\circ$), but the trigonometric functions are defined for any value of x.

These are *periodic functions* with a period of 360°(or 2π radians) so we can focus on their definition on the domain $0^\circ \leq x \leq 360^\circ$, for example, $\sin a = \sin(a + 2\pi n)$, where n is an integer.

Looking at the *unit circle* - a circle with a radius of a 1 centered at the origin of a two-dimensional Cartesian coordinate system as Illustrated in Fig. 2.



Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degree	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Figure 2: Unit circle illustration and simple algebraic values.

The radius that creating an angle θ with the x axis is meeting the circle at the point $(\cos\theta, \sin\theta)$.
The conversion between radians and degrees is given by

$$Radians = Degrees \times \frac{\pi}{180}.$$

Exercise:

1. Convert the following angles from radians to degrees:
   (a) $\frac{\pi}{12}$
   (b) $\frac{\pi}{8}$
   (c) $\frac{5\pi}{12}$
   (d) 0.5
   (e) 1
2. Convert the following angles from degrees to radians:
   (a) $135°$
   (b) $22.5°$
   (c) $67.5°$
   (d) $10°$
   (e) $100°$
3. Use the unit circle to express the following expressions using $sin(x)$ and $cos(x)$:
   (a) $\sin(-x) = ?$
   (b) $\sin(180 - x) = ?$
   (c) $\cos(-x) = ?$
   (d) $\cos(180 - x) = ?$
4. Search online for the graphs of the trigonometric functions and compare between the graphs:
   (a) Find the values of x where the functions meet the x axis.
   (b) What is the maximal\minimal value for each function?

2 Exponential and Power Laws

Exponentiation is a mathematical operation written as:

$$b^n = b \times b \times \cdots (\text{n times}),$$

where n is the power (or exponent) and b is the base.
power laws:

1. $b^n b^m = b^{n+m}$
2. $(b^n)^m = b^{n*m}$
3. $b^{1/n} = \sqrt[n]{b}$
4. $b^0 = 1$, for any value of b.

Logarithm is the inverse operation to exponentiation:
if $y = \log_b x$ it means that $x = b^y$.
moreover, the following holds: $\log_b b^n = n$ and $b^{\log_b n} = n$.
The functions $e^x$and $\ln x$:
e is a mathematical constant roughly equal to 2.718. $\ln x = \log_e x$ is the inverse function to e^x.
it follows that:

1. $\ln e = 1$
2. $\ln x + \ln y = \ln (x \times y)$
3. $\ln x^n = n \ln x$

3 Euler's Formula

Euler's formula states that a *complex number* e^{ix} follows:

$$e^{ix} = \underbrace{\cos x}_{Re} + i \underbrace{\sin x}_{Im}, \tag{3.1}$$

where $i = \sqrt{-1}$ is the imaginary unit and the complex number is divided into a real part (Re) and and imaginary part (Im).
This formula describes the relation between the trigonometric functions and the complex exponential function.
The trigonometric functions can be written in term of the exponential function:

$$\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2} \end{aligned} \tag{3.2}$$

Exercise:
Calculate: $e^{i0} =?$, $e^{i\pi} =?$, $e^{\frac{i\pi}{2}} =?$, $e^{\frac{i3\pi}{4}} =?$.

4 Differentiation

The derivative of f with respect to x is a measure of the rate of change of the value of the function $f(x)$ with respect to the change of the variable x.
A mathematical definition:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4.1}$$

Basic Rules:

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(C) = 0$, where C is a constant.
3. $\frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$
4. $\frac{d}{dx}(f[g(x)]) = \frac{df}{dg}\frac{dg}{dx}$
5. $\frac{d}{dx}(f \times g) = \frac{df}{dx}g + f\frac{dg}{dx}$
6. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right)$
7. $\frac{d}{dx}(\ln[f(x)]) = \frac{1}{f(x)}\frac{df}{dx}$
8. $\frac{d}{dx}\left(e^{f(x)}\right) = \frac{df}{dx}e^{f(x)}$

Exercise:
Calculate:

1. $4x^2 + 3x + 1$
2. $\frac{5}{x}$
3. $(1 + x)(1 - 7x)$
4. $\frac{x^2}{3x-1}$
5. $(8 - x)^3$
6. $\ln(2x)$
7. $4e^x$

5 Integrals

Integral is the inverse operation to differentiation, up to an additive constant ($\frac{d}{dx}(\text{const}) = 0$).
Indefinite integral:

$$\int f(x)dx = F(x) + C \tag{5.1}$$

Where, $F'(x) = \frac{dF}{dx} = f(x)$.
Definite integral:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \tag{5.2}$$

Basic Rules:

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
2. $\int C dx = Cx + D$
3. $\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\ dx + \beta \int g(x) dx$
4. $\int e^x\,dx = e^x + C$
5. $\int \frac{1}{x}\,dx = \ln(x) + C$
6. $\int_a^b f\,dx = -\int_b^a f\,dx$

6 Vectors

Vectors are geometric representations of magnitude and direction which are often represented by straight arrows, starting at one point on a coordinate axis and ending at a different point. That is, in d dimensions any vector is described by d numbers, for example: in 3 dimensions one may describe a vector with the value of its magnitude and 2 angles that describe its direction - total of 3 numbers. Vectors are crucial to physics, since a large portion of physical quantities have both magnitude and direction (distance, velocity, acceleration, momentum, etc.), using vectors is often more intuitive and simplifies calculations.

Multiplication by scalar: Multiplying a vector $\boldsymbol{A} = (A_1, A_2, A_3 \dots)$, sometimes denoted also as $\boldsymbol{A} = (A_x, A_y, A_z \dots)$, by a scalar (a quantity with magnitude alone, without direction: constants, functions, etc.) α result in a vector with the same direction and different magnitude $\alpha\boldsymbol{A} = (\alpha A_1, \alpha A_2, \alpha A_3 \dots)$ as demonstrated in Fig. 3.

Addition: A vector can be also represented by its components. In *Cartesian coordinate system*, for example, the components of a vector can be found by subtracting the "initial" coordinates from the "final" coordinates: in Fig. 4 the components of vector $\boldsymbol{B}$ (the arrow

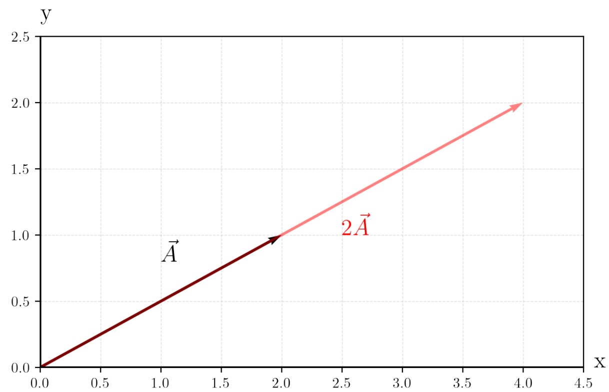


Figure 3: Vector $\boldsymbol{v} = (2, 1)$ (black) multiplied by a constant scalar 2 result in $2\boldsymbol{v} = (4, 2)$ (red) which is "twice as long", this is true for any scalar (not necessarily a constant).

originated at point a and terminates at point b), which is also denoted by $\vec{ab}$ or $\vec{B}$, are (the same for the red and black in Fig. 4)

$$B_1 = 3 - 2 = 1 - 0 = 1,$$
$$B_2 = -2 - 3 = -5 - 0 = -5,$$

where the subscript B_1 denotes the x component of vector $\boldsymbol{B}$. Vectors can be added by summing on each component,

$$\boldsymbol{A} + \boldsymbol{B} \quad \rightarrow \quad \begin{cases} (\boldsymbol{A} + \boldsymbol{B})_1 = A_1 + B_1 \\ (\boldsymbol{A} + \boldsymbol{B})_2 = A_2 + B_2. \end{cases} \tag{6.1}$$

In Fig 4, for example,

$$(\boldsymbol{A} + \boldsymbol{B})_1 = 2 + 1 = 3,$$
$$(\boldsymbol{A} + \boldsymbol{B})_2 = 3 + (-5) = -2.$$

Note that the sum $\boldsymbol{A} + \boldsymbol{B}$ yields a new vector which connects the initial and final points (placing the origin of one vector at the end of the other).

Norm: the magnitude of a vector, a.k.a. *norm*, can be easily expressed using geometry. Looking at Fig. 5, it is clear that the length of $\boldsymbol{A}$ is $\sqrt{A_1^2 + A_2^2}$. This magnitude, or the norm of $\boldsymbol{A}$, can be generalized to any number of dimensions,

$$|\boldsymbol{A}| \equiv \sqrt{A_1^2 + A_2^2 + A_3^2 + \cdots}. \tag{6.2}$$

Unit vectors: A unit vector, denoted by $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ (sometimes $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, or $\boldsymbol{e}_i$ with $i = 1, 2, 3 \dots$), is a vector with magnitude of unity, which means it describes only a direction (since its

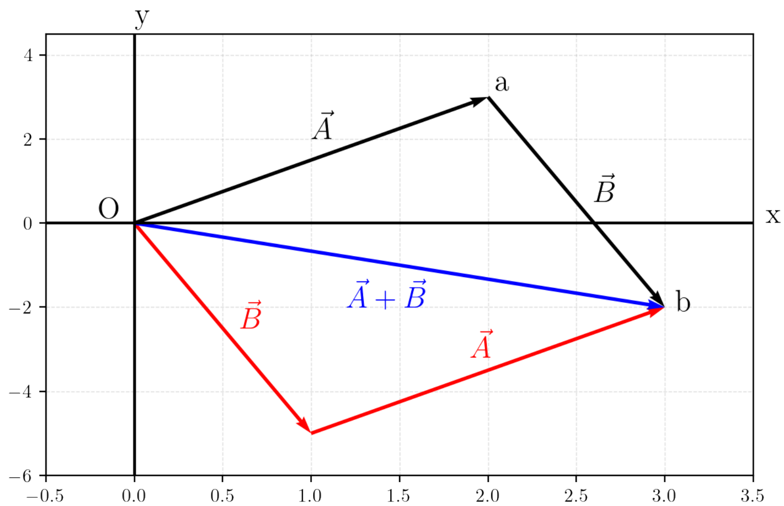


Figure 4: Vectors $\boldsymbol{u} = \vec{Oa}$, $\boldsymbol{v} = \vec{ab}$ (black and red) and $\boldsymbol{A} + \boldsymbol{B} = \vec{Ob}$ (blue).

magnitude is already known to be 1); for example, a set of orthogonal unit vectors (also called orthonormal basis, explained below) can be defined as

$$\begin{aligned} \hat{\boldsymbol{i}} &= \hat{\boldsymbol{x}} = \boldsymbol{e}_1 = (1, 0, 0, \dots) \\ \hat{\boldsymbol{j}} &= \hat{\boldsymbol{y}} = \boldsymbol{e}_2 = (0, 1, 0, \dots) \\ \hat{\boldsymbol{k}} &= \hat{\boldsymbol{z}} = \boldsymbol{e}_3 = (0, 0, 1, \dots) \end{aligned} \tag{6.3}$$

Therefore, a vector $\boldsymbol{v}$ can be expressed algebraically simply by summing over *unit vectors* multiplied by scalars

$$\begin{aligned} \boldsymbol{A} = (A_1, A_2, A_3 \dots) &= A_1\hat{\boldsymbol{i}} + A_2\hat{\boldsymbol{j}} + A_3\hat{\boldsymbol{k}} + \cdots \\ &= A_1\hat{\boldsymbol{x}} + A_2\hat{\boldsymbol{y}} + A_3\hat{\boldsymbol{z}} + \cdots \\ &= A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3 + \cdots , \end{aligned} \tag{6.4}$$

as shown in Fig. 5.

Scalar product (dot product): another useful operation with vectors is the scalar product, which takes 2 vectors and returns a scalar. It is denoted by "$\cdot$" and defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} \equiv |\boldsymbol{A}|\,|\boldsymbol{B}| \cos\theta = A_1B_1 + A_2B_2 + A_3B_3 + \cdots \tag{6.5}$$

where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, as shown in Fig. 6. Looking at (6.3) we find that the scalar product of orthogonal vectors vanishes: $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = 0$.
A notion of meaning of a scalar product can be intuitively understood when applied to a unit vector, for example: consider the scalar product of $\boldsymbol{A} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}}$ and the unit vector

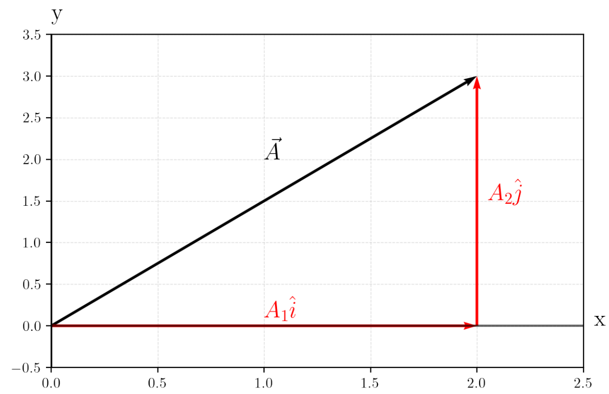


Figure 5: Vector $\boldsymbol{A} \equiv \vec{A} = A_1\hat{\boldsymbol{\imath}} + A_2\hat{\boldsymbol{\jmath}}$ represented by a sum of unit vectors.

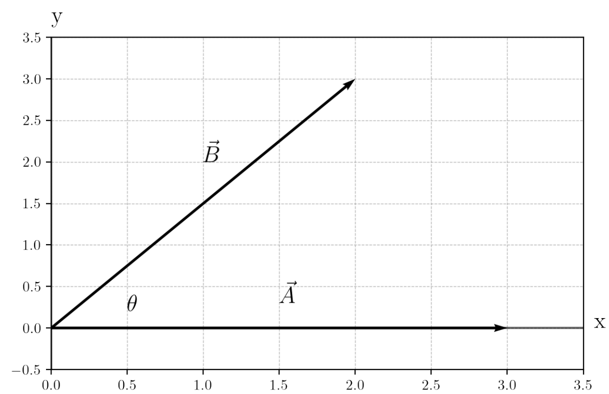


Figure 6: Scalar product can be written as $\boldsymbol{A} \cdot \boldsymbol{B} \equiv |\boldsymbol{A}|\,|\boldsymbol{B}| \cos\theta$.

$\hat{\imath}$, using (6.5)

$$\boldsymbol{A} \cdot \hat{\boldsymbol{\imath}} = \sqrt{A_1^2 + A_2^2}\cos\theta = A_1$$

or

$$\boldsymbol{A} \cdot \hat{\boldsymbol{\imath}} = A_1\left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}\right) + A_2\cancel{\left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}\right)} = A_1$$

where $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \left|\hat{\boldsymbol{\imath}}\right|^2 = 1$ since it has magnitude 1. Thus, the scalar product with unit vectors returns the component of the vector $\boldsymbol{v}$ in the direction of the unit vector (or the magnitude of the projection of $\boldsymbol{A}$ in direction $\hat{\imath}$).
Therefore, if we want to find the component of a vector $\boldsymbol{A}$ in the direction of another vector $\boldsymbol{B}$, all we need to do is to take the scalar product of $\boldsymbol{A}$ with a unit vector $\hat{\boldsymbol{B}}$ (unit vector in

the direction of $\boldsymbol{B}$)

$$A_{\hat{\boldsymbol{B}}} = \boldsymbol{A} \cdot \hat{\boldsymbol{B}} = \boldsymbol{A} \cdot \frac{\boldsymbol{B}}{|\boldsymbol{B}|}.$$

Vector product (cross product): a cross product, denoted by "$\times$", takes 2 vectors and returns a new vector, which is orthogonal to the original vectors. It is defined as

$$\boldsymbol{A} \times \boldsymbol{B} \equiv -\boldsymbol{B} \times \boldsymbol{A} \equiv |\boldsymbol{A}|\,|\boldsymbol{B}| \sin\theta\, \hat{\boldsymbol{n}}, \tag{6.6}$$

so that $\hat{\boldsymbol{B}} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{A}} \cdot \hat{\boldsymbol{n}} = 0$, where $\hat{\boldsymbol{n}}$ is the direction of the new vector, which is determined according to the *right-hand-rule* illustrated in Fig. 7.

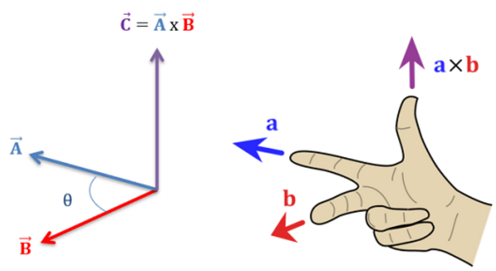


Figure 7: The right-hand-rule which determined the direction of the $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$.

An expression which involves the components of the vectors is

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{B} \equiv & \,(A_2B_3 - A_3B_2)\,\hat{\boldsymbol{\imath}} \\ & + (A_3B_1 - A_1B_3)\,\hat{\boldsymbol{\jmath}} \\ & + (A_1B_2 - A_2B_1)\,\hat{\boldsymbol{k}}. \end{aligned} \tag{6.7}$$

The intuitive explanation of the cross product is easy to read from (6.6) as $|\boldsymbol{A}|\,|\boldsymbol{B}| \sin\theta$ is the area of a parallelogram $a = h \times b$ (Fig. 8) with height $h = |\boldsymbol{B}| \sin\theta$ and base $b = |\boldsymbol{A}|$. The direction $\hat{\boldsymbol{n}}$ is perpendicular to the surface of the parallelogram and chosen (there are 2 sides to the surface) according to the right-hand-rule.

Derivatives and integration: To take a derivative or integrate a vector $\boldsymbol{A}$ with respect to time t one should consider each component separately (we assume that the unit vectors do not vary with time):

$$\frac{d\boldsymbol{A}}{dt} = \frac{dA_1}{dt}\hat{\boldsymbol{\imath}} + \frac{dA_2}{dt}\hat{\boldsymbol{\jmath}} + \frac{dA_3}{dt}\hat{\boldsymbol{k}},$$

$$\int \boldsymbol{A} dt = \int A_1 dt \hat{\boldsymbol{\imath}} + \int A_2 dt \hat{\boldsymbol{\jmath}} + \int A_3 dt \hat{\boldsymbol{k}}.$$

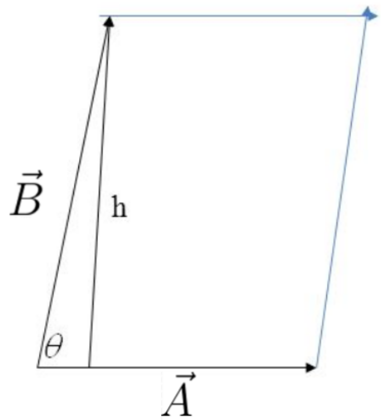


Figure 8: The area of parallelogram built from vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is $a = h\,|\boldsymbol{A}| = |\boldsymbol{B}|\sin\theta\;|\boldsymbol{A}|$.

<u>**Summary:**</u>

- *Norm*: the magnitude of a vector $\boldsymbol{A}$ defined as $|\boldsymbol{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$.
- *Multiplication by scalar*: $\alpha\boldsymbol{A} = \alpha A_1\hat{\boldsymbol{\imath}} + \alpha A_2\hat{\boldsymbol{\jmath}} + \alpha A_3\hat{\boldsymbol{k}}$.
- *Addition*: $\boldsymbol{A} + \boldsymbol{B} \equiv (A_1 + B_1)\,\hat{\boldsymbol{\imath}} + (A_2 + B_2)\,\hat{\boldsymbol{\jmath}} + (A_3 + B_3)\,\hat{\boldsymbol{k}}$.
- *Unit vectors*: any vector $\boldsymbol{A}$ can be associated with a unit vector $\hat{\boldsymbol{A}}$ that describes its direction $\hat{\boldsymbol{A}} = \frac{\boldsymbol{A}}{|\boldsymbol{A}|}$.
- *Scalar product*: $\boldsymbol{A}\cdot\boldsymbol{B} \equiv |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta = A_1B_1 + A_2B_2 + A_3B_3$.
- *Vector product*: $\boldsymbol{A}\times\boldsymbol{B} \equiv -\boldsymbol{A}\times\boldsymbol{B} \equiv |\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is determined by the right-hand-rule.
- *Time derivative*: $\frac{d\boldsymbol{A}}{dt} = \frac{dA_1}{dt}\hat{\boldsymbol{\imath}} + \frac{dA_2}{dt}\hat{\boldsymbol{\jmath}} + \frac{dA_3}{dt}\hat{\boldsymbol{k}}$.
- *Time integration*: $\int \boldsymbol{A}dt = \int A_1dt\hat{\boldsymbol{\imath}} + \int A_2dt\hat{\boldsymbol{\jmath}} + \int A_3dt\hat{\boldsymbol{k}}$.

Additional reading: Kleppner Sec. 1.1-1.7.

<u>**Exercise:**</u>

Using vectors $\boldsymbol{A} = (1, -4, 2)$ (in Cartesian coordinate system) and $\boldsymbol{B} = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$:

1. Calculate the vector $\boldsymbol{C} = \boldsymbol{A} + 2\boldsymbol{B}$, draw it.
2. Calculate the vector $\boldsymbol{D} = 2\boldsymbol{A} + 4\boldsymbol{B}$, draw it.
3. Calculate the norm $|\boldsymbol{C}|$ and its angle with $\hat{\boldsymbol{\jmath}}$ axis, φ.
4. Calculate the scalar product of $\boldsymbol{C}\cdot\boldsymbol{B}$, use both expressions in (6.5).
5. Calculate the unit vector $\hat{\boldsymbol{C}}$.

6. The cross product $\boldsymbol{C} \times \boldsymbol{D}$, use both (6.6) and (6.7). What is the direction of the resulting vector?

7. Take the derivative of the vector $\boldsymbol{A}(t) = t^2\boldsymbol{e}_1 + \frac{1}{t}\boldsymbol{e}_2$.

8. Integrate over the vector $\boldsymbol{A}(t) = t^2\hat{\boldsymbol{x}} + \frac{1}{t}\hat{\boldsymbol{y}}$ from $t_0 = 3$ to $t_1 = 5$.

7 Polar Coordinates

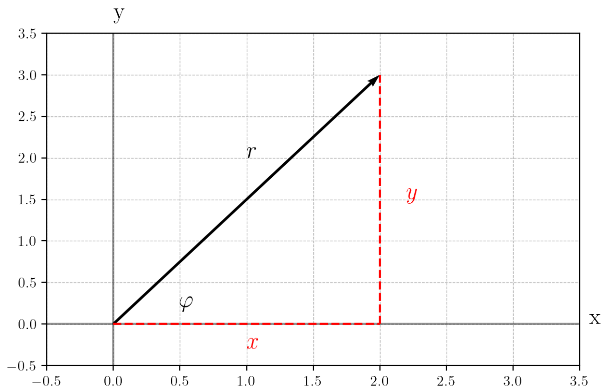


Figure 9: Illustration of polar coordinates.

Sometimes a problem has a symmetry which makes it easier to describe by a different set of coordinates than Cartesian. One useful set of coordinates is *polar coordinates* which describe any point on 2 dimensional space by the distance from the origin r and an angle φ (usually taken with respect to the x axis), instead of the x and y Cartesian coordinates. The relations between Cartesian and polar coordinates can be deduced with simple geometry as, shown in Fig. 9,

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \longleftrightarrow \begin{cases} r &= \sqrt{x^2+y^2} \\ \varphi &= \tan^{-1}\frac{y}{x} \end{cases} \tag{7.1}$$

Relation to vectors: In order to describe a vector $\boldsymbol{v}$ in polar coordinates as $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\varphi\hat{\boldsymbol{\varphi}}$ one relate Cartesian and polar unit vectors

$$\begin{aligned} \hat{\boldsymbol{r}} &= \cos\varphi\hat{\boldsymbol{x}} + \sin\varphi\hat{\boldsymbol{y}}, \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}}. \end{aligned} \tag{7.2}$$

Time derivatives: Since polar coordinates may vary with time, since they change their direction at different positions in space, one should treat the time derivative of a vector more carefully. For example, in physics we often use the position vector $\boldsymbol{r}$ to describe the position of a particle (or any point like object), in two dimensions

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} = r\hat{\boldsymbol{r}}.$$

It is easy to see why polar coordinates offer a more compact description of $\boldsymbol{r}$. Since $\hat{\boldsymbol{r}}$ points to different directions for different points in space then when we take the time derivative of the position vector we find

$$\frac{\partial\boldsymbol{r}}{\partial t} = \frac{\partial r}{\partial t}\hat{\boldsymbol{r}} + r\frac{\partial\hat{\boldsymbol{r}}}{\partial t}.$$

We may calculate the time derivative $\partial\hat{\boldsymbol{r}}/\partial t$ by using (7.2),

$$\frac{\partial\hat{\boldsymbol{r}}}{\partial t}=\frac{\partial}{\partial t}\left(\cos\varphi\hat{\boldsymbol{x}}+\sin\varphi\hat{\boldsymbol{y}}\right)=\frac{\partial\varphi}{\partial t}\underbrace{\left(-\sin\varphi\hat{\boldsymbol{x}}+\cos\varphi\hat{\boldsymbol{y}}\right)}_{\hat{\boldsymbol{\varphi}}}=\frac{\partial\varphi}{\partial t}\hat{\boldsymbol{\varphi}},$$

thus

$$\frac{\partial\boldsymbol{r}}{\partial t}=\frac{\partial r}{\partial t}\hat{\boldsymbol{r}}+r\frac{\partial\varphi}{\partial t}\hat{\boldsymbol{\varphi}}.$$

As we take time derivatives we must always consider the derivatives of both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\varphi}}$.

Summary:

- *Polar coordinates definition*:

$$\begin{matrix} x=r\cos\varphi \\ y=r\sin\varphi \end{matrix} \quad\longleftrightarrow\quad \begin{matrix} r=\sqrt{x^2+y^2} \\ \varphi=\tan^{-1}\frac{y}{x} \end{matrix}$$

- *Unit vectors*:

$$\begin{aligned}\hat{\boldsymbol{r}}&=\cos\varphi\hat{\boldsymbol{x}}+\sin\varphi\hat{\boldsymbol{y}},\\ \hat{\boldsymbol{\varphi}}&=-\sin\varphi\hat{\boldsymbol{x}}+\cos\varphi\hat{\boldsymbol{y}}.\end{aligned}$$

- *Time derivatives*: As we take time derivatives of a vector represented in polar coordinates we must always consider the time derivatives of both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\varphi}}$ which vary from point to point.

- Later on we will encounter more coordinate systems such as *spherical* and *cylindrical* coordinates.

Additional reading: Kleppner Sec. 1.11.1-1.11.2.

Exercise:

Using the vector $\boldsymbol{r}(t)=-3t\hat{\boldsymbol{i}}+\left(5t^2-1\right)\hat{\boldsymbol{j}}$:

1. Calculate its polar components r and φ.
2. Write it in terms of polar unit vectors.
3. take the time derivative of $\boldsymbol{r}(t)$ using both polar and Cartesian representations. Did you get the same result? (try plugging in some values for t if you can't tell right away)
4. Calculate the scalar product of $\boldsymbol{A}=(2,0.4\pi)$ and $\boldsymbol{B}=(3,1.3\pi)$, given in polar coordinates.

8 Sketching Graphs

This is one of the most important tools to understand the behavior of functions and anticipating the outcome of many calculations. Here is a set of steps to follow:

1. What is the *domain*? $-\infty < x < \infty$, $0 < x < L$, $-\pi < \varphi < \pi$ etc.

2. Plot *obvious points* that you can identify immediately.

3. Is the function *even or odd* $[f(x) = \pm f(-x)]$? If so, concentrate on understanding one side and then make a mirror image on the other.

4. Is the function *singular* anywhere? If there are points in which the function goes to $\pm\infty$ (if the denominator vanishes for example) make a note for them.

5. What is the *behavior* of the function *near* any of the obvious points that you plotted? Does it behave like x? Like x^2? etc.

6. Characterize the singular points: how does the function behaves (goes to $+\infty$ or $-\infty$) as you approach it from the left and right?

7. How does the function behaves at the boundaries of the domain?

8. Is it possible to separate the function to two much *simpler functions*? For example, a sum or product of two simpler functions.

9. Is the function related to another by translation? For example, $f(x) = (x-2)^2$ is related to x^2 by translation of 2 units.

10. *Interpolate* the behavior between the points you've found.

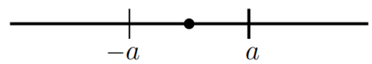
Example: $f(x) = \frac{x}{a^2 - x^2}$.

(1) The domain is not specified, thus we take $-\infty < x < \infty$.

(2) The point $x = 0$ obviously gives $f(0) = 0$.

(4) The denominator becomes zero at the two points $x = \pm a$.

Therefore we mark the points:

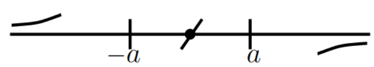


(3) Replacing x by $-x$ changes the sign of the numerator but does not affect the denominator, thus the function is odd about zero.

(7) When x becomes very large ($|x| \gg a$), the denominator is mostly $-x^2$, so $f(x)$ behaves like $\frac{x}{-x^2} = -\frac{1}{x}$, i.e. it approaches zero for large x. Moreover, when $x > 0$ approaches zero through negative values and when $x < 0$, it goes to zero through positive values.

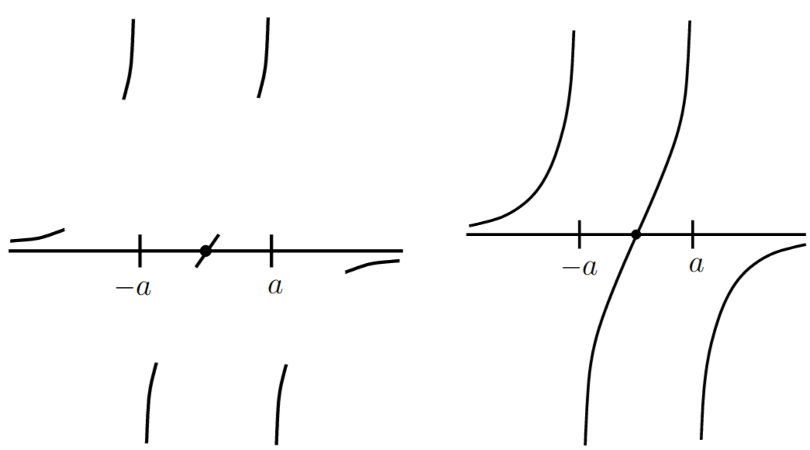
(5) When x is very small ($|x| \ll a$), the denominator is mostly a^2. That means that near $x = 0$, the function f behaves like $\frac{x}{a^2}$ (linear).

Therefore we note the behavior near those point:



(6) Investigating the singular points $\pm a$ we see that: if $x \gtrsim a$, the x^2 in the denominator is slightly larger than the a^2. This means that the denominator is negative. When $x \lesssim a$, the reverse is true. Therefore the function approaches $-\infty$ as $x \to a$ from the right, while it approaches $+\infty$ on the left side of a. Since the function is odd we do not need to repeat this analysis near $x = -a$, only to turn this behavior upside down.

(10) Interpolate to see the whole picture.



We could have realized that f can be rearranged as

$$\frac{x}{a^2 - x^2} = \frac{-1/2}{x - a} + \frac{-1/2}{x + a},$$

and then follow the ideas of techniques (8) and (9) to sketch the graph.

Exercise:
Sketch the functions (set the constants v_0, m, g,ℓ, V_0, a, x_0 and ω to 1):

1. $y(t) = v_0 t - gt^2/2$.
2. $U(y) = -mgy + ky^2/2$.
3. $U(\theta) = mg\ell(1 - \cos\theta)$.
4. $V(x) = V_0 e^{-x^2/a^2}$.
5. $x(t) = x_0 e^{-\alpha t} \sin(\omega t)$