## HOME EX. 1: FOURIER RECAP

We define the Fourier transform of an $a$-preiodic function $f(x) = f(x+a)$ by:

$$\begin{aligned} f(x) &= \sum_k f_k e^{ikx} \\ f_k &= \frac{1}{a}\int_{-a/2}^{a/2} f(x)e^{-ikx}dx \end{aligned} \tag{1}$$

If the function $f(x)$ has infinite period, then it can be defined as follows:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{ikx}dk \\ f(k) &= \int_{\infty}^{\infty} f(x)e^{-ikx}dx \end{aligned} \tag{2}$$

**1.**
What is the fourier transform $U(k)$ of the function:

$$U(x) = U_0 a\left[2\delta(x-na) - \delta(x-(n+1/4)a) - \delta(x-(n+3/4)a)\right] \tag{3}$$

With $n \in [-\infty, .., -1, 0, .., \infty]$

(Taken from Exam 2011 MOED A)

**Solution:** We note that this is $a$-periodic, and also symmetric around $x = 0$. The transform is simple:

$$f(k) = \frac{1}{a}\int_{-a/2}^{a/2} U_0 a[-\delta(x+\frac{a}{4}) + 2\delta(x) - \delta(x-\frac{a}{4})]e^{ikx}dx = 2U_0[1-\cos(ka/4)] \tag{4}$$

$$f(x)/U_0 = \sum_{k>0} 4[1-\cos(ka/4)]\cos(kx) = \sum_{n=1,2,..} 4[1-\cos(\pi n/2)]\cos(2\pi nx/a) \tag{5}$$

**2.**
Define $f(x) = x$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$.
Find a function $\tilde{f}(x) = \sum_k \tilde{f}(k)e^{ikx}$, so that $\tilde{f}(x) = f(x)$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$. What are the allowed values of $k$? draw $\tilde{f}(x)$.
**Note:** If f(x) is not continous in $x_0$, then its fourier representation will give $\frac{1}{2}[f(x_0-) + f(x_0+)]$ in $x = x_0$.

**Solution:** We need to expand $f(x)$, otherwise the fourier series will converge to zero at $x = \pm a/2$. One option is to define $\tilde{f}(x)$ with $-a < x < a$:

$$\tilde{f}(x) = \begin{cases} -a-x & , x < a/2 \\ x & , -a/2 < x < a/2 \\ a-x & , x > a/2 \end{cases} \tag{6}$$

With $\tilde{f}(x)$ $2a$-periodic. Since the function is asymmetric (and real), the fourier series takes the form:

$$\tilde{f}(x) = \sum_{n=1}^{\infty} \tilde{f}_n \sin(\pi nx/a) \tag{7}$$

$$\tilde{f}_n = \frac{2}{a}\int_0^a dx\tilde{f}(x)\sin(\pi nx/a) = \frac{4\sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2} \tag{8}$$

**3.** Dirac's Comb and diffraction

**Solution:**
**a)** $\Delta(x)$ is periodic with period $a$. Its fourier transform $f(k)$ only have components with $k_n = (2\pi n)/L$. $f(k) \propto \sum_n \delta(k - k_n)$. $\Delta(k)$ is then peaked at $k_n$ and zero otherwise.

**b)**
For a finite comb we have:

$$\Delta_N(k) = e^{-ik(N/2)} \sum_{n=0}^{N} e^{ikan} = e^{-ik(N/2)} \frac{1 - x^{N+1}}{1 - x} \tag{9}$$

With $x = \exp(ika)$.

**c)**
The maximum value of $V_N(k)$ is attained when all the terms in $\Delta_N(k)$ have the same phase. This happens when the phase difference of successive terms is $2\pi$, that is $ka = 2\pi$. The maximum value is then $V_N \approx N^2$.

**d)**
We have:

$$\frac{V_N(k)}{N^2} = \frac{1}{N^2} \left[ \frac{\sin(kaN/2)}{\sin(ka/2)} \right]^2 \tag{10}$$

This is an oscillatory function of $k$, with the maximum value of 1 at $k = 2\pi n/a$. It goes to zero for $k = 2\pi n/(Na)$. For large $N$ it goes to zero wherever $k \neq 2\pi/a$. See Fig. 1

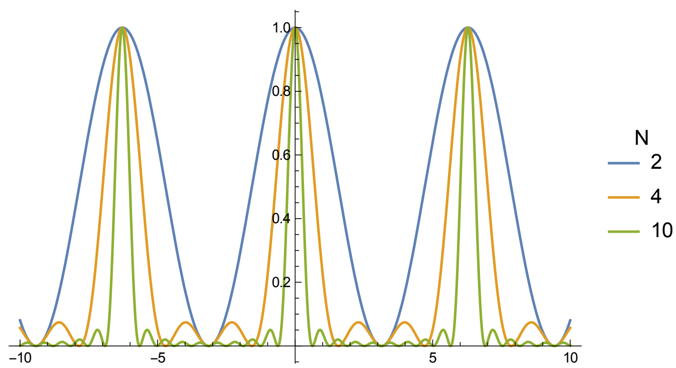


FIG. 1. $V_N(k)$ for different values of $N$