Selected topics in solid state physics 2

Part 5

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(Dated: December 15, 2021)

I. PATH INTEGRAL IN FIELD THEORY

A. Single oscillator, bosonic representation

Here I closely follow the book of Kamenev [1]. We consider a single harmonic oscillator described by the Hamiltonian $H = \hbar\omega_0 b^\dagger b$. One uses the concept of coherent states, which are eigenstates of b. A coherent states is defined as

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} \, |n\rangle \ , \tag{1}$$

where ϕ is a complex number. Using $b\,|n\rangle = \sqrt{n}\,|n\rangle$, it is easy to see that

$$b\,|\phi\rangle = \phi\,|\phi\rangle \ . \tag{2}$$

The coherent states are not mutually orthogonal

$$\langle \phi_2 | \, \phi_1 \rangle = \sum_{n=0}^{\infty} \frac{\bar{\phi}_2^n \phi_1^n}{n!} = \exp\left[\bar{\phi}_2 \phi_1\right] \ . \tag{3}$$

We use the notation $\bar{\phi} = \phi^*$ for complex conjugation. The basis of coherent states is over-complete. Yet, it is possible to introduce the resolution of unity as

$$\hat{1} = \int \frac{d\mathrm{Re}\phi \, d\mathrm{Im}\phi}{\pi} \, e^{-\bar{\phi}\phi} \, |\phi\rangle \, \langle\phi| = \int d[\bar{\phi}, \phi] \, e^{-\bar{\phi}\phi} \, |\phi\rangle \, \langle\phi| \ . \tag{4}$$

For brevity one uses the notation

$$d[\bar{\phi}, \phi] \equiv \frac{d\mathrm{Re}\phi \, d\mathrm{Im}\phi}{\pi} \ . \tag{5}$$

To prove (4) it is useful to go to circular coordinates $\mathrm{Re}\phi = r\cos\theta$, $\mathrm{Im}\phi = r\sin\theta$, so that $d[\bar{\phi}, \phi] = (1/\pi) r dr d\theta$.

We now pursue a seemingly useless task of rewriting a number 1 in a very complicated way. We consider the oscillator as a many-body system with $\hat{N} = b^\dagger b$ being the number of particles. We prepare the oscillator in a thermal equilibrium state

$$\rho_0 = \frac{e^{-\beta(H-\mu N)}}{\mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right]} = \frac{1}{\mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right]} \, e^{-\beta(\hbar\omega_0 - \mu) b^\dagger b} \ . \tag{6}$$

For the denominator we obtain

$$\mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right] = \sum_{n=0}^{\infty} e^{-\beta(\hbar\omega_0 - \mu)n} = \frac{1}{1 - e^{-\beta(\hbar\omega_0 - \mu)}} \ . \tag{7}$$

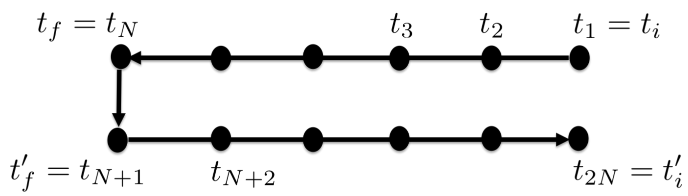

FIG. 1: Keldysh contour with discrete times.

We propagate the oscillator's state in time and obtain the trivial result

$$1 = \mathrm{Tr}\left[e^{-(i/\hbar)Ht}\,\rho_0\,e^{(i/\hbar)Ht}\right] \ . \tag{8}$$

We try to rewrite this with the help of the path integral. We use again the Keldysh contour as shown in Fig. 1 and obtain

$$1 = \mathrm{Tr}\left[\prod_{n=1}^{N-1} e^{-(i/\hbar)(t_{n+1}-t_n)H}\,\rho_0 \prod_{n=N+1}^{2N-1} e^{(i/\hbar)(t_{n+1}-t_n)H}\right] \ . \tag{9}$$

Now we insert resolutions of unity (4) at each t_n and obtain

$$1 = \int \prod_{n=1}^{2N} d[\bar{\phi}_n, \phi_n]\ \mathrm{Tr}\,\Big[$$
$$|\phi_N\rangle \ldots \langle\phi_2|\, e^{-(i/\hbar)H\Delta t}\,|\phi_1\rangle\, e^{-\bar{\phi}_1\phi_1}\,\langle\phi_1|\,\rho_o\,|\phi_{2N}\rangle\, e^{-\bar{\phi}_{2N}\phi_{2N}}\,\langle\phi_{2N}|\, e^{(i/\hbar)H\Delta t}\,|\phi_{2N-1}\rangle \ldots .\,\langle\phi_{N+1}|\Big] \ . \tag{10}$$

Useful identity. We now use the following identity

$$f(x) = \langle\phi_2|\, e^{\ln x\, b^\dagger b}\,|\phi_1\rangle = e^{x\,\bar{\phi}_2\phi_1} \ . \tag{11}$$

Indeed

$$\begin{aligned}\frac{d}{dx} f(x) &= \langle\phi_2|\, b^\dagger b(1/x) e^{\ln x\, b^\dagger b}\,|\phi_1\rangle = \langle\phi_2|\, b^\dagger b e^{\ln x\,(b^\dagger b - 1)}\,|\phi_1\rangle \\ &= \langle\phi_2|\, b^\dagger e^{\ln x\, b^\dagger b}\, b\,|\phi_1\rangle = \bar{\phi}_2\phi_1 f(x) \ .\end{aligned} \tag{12}$$

Since $f(0) = e^{\bar{\phi}_2\phi_1}$ we obtain the desired identity. Thus we obtain

$$\langle\phi_1|\,\rho_o\,|\phi_{2N}\rangle = (1 - e^{-\beta(\hbar\omega_0-\mu)})\ \langle\phi_1|\, e^{-\beta(\hbar\omega_0-\mu)b^\dagger b}\,|\phi_{2N}\rangle = (1-\chi_0)\, e^{\chi_0\bar{\phi}_1\phi_{2N}} \ , \tag{13}$$

where $\chi_0 \equiv e^{-\beta(\hbar\omega_0-\mu)}$.

We use, first, $\mathrm{Tr}\left[|a\rangle\langle b|\right] = \langle b|\, a\rangle$. Thus we obtain

$$1 = \int \prod_{n=1}^{2N} d[\phi_n, \phi_n]$$
$$e^{-\bar{\phi}_{2N}\phi_{2N}} \langle \phi_{2N}| e^{(i/\hbar)H\Delta t} |\phi_{2N-1}\rangle \ldots . \langle \phi_{N+1}| \hat{1} |\phi_N\rangle \ldots \langle \phi_2| e^{-(i/\hbar)H\Delta t} |\phi_1\rangle e^{-\bar{\phi}_1\phi_1} \langle \phi_1| \rho_o |\phi_{2N}\rangle \ . \tag{14}$$

Next we approximate

$$\langle \phi_{n+1}| e^{-(i/\hbar)H(b^\dagger b)\Delta t} |\phi_n\rangle \approx \langle \phi_{n+1}| \left[1 - (i/\hbar)H(b^\dagger b)\Delta t\right] |\phi_n\rangle$$
$$= \langle \phi_{n+1}|\,|\phi_n\rangle \left[1 - (i/\hbar)H(\bar{\phi}_{n+1}\phi_n)\Delta t\right] \approx e^{\bar{\phi}_{n+1}\phi_n} e^{-(i/\hbar)H(\bar{\phi}_{n+1}\phi_n)\Delta t} \ . \tag{15}$$

This is so for any normal-ordered Hamiltonian. All in all we obtain

$$1 = (1 - \chi_0) \int \prod_{k=1}^{2N} d[\bar{\phi}_k, \phi_k] \, \exp\left[i\mathcal{S}\right] \ , \tag{16}$$

where

$$i\mathcal{S} = \sum_{n,m} \bar{\phi}_m (iG^{-1})_{mn} \phi_n \tag{17}$$

and the matrix iG^{-1} looks like

$$iG^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \chi_{\mathbf{0}} \\ h_- & -1 & 0 & 0 & 0 & 0 \\ 0 & h_- & -1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 0 \\ 0 & 0 & 0 & h_+ & -1 & 0 \\ 0 & 0 & 0 & 0 & h_+ & -1 \end{pmatrix} \ . \tag{18}$$

Here $h_\mp = 1 \mp i\omega_0\Delta t$ and we took the example of $N = 3$. We rewrite

$$i\mathcal{S} = \sum_{j=2}^{2N} \bar{\phi}_j(\phi_{j-1} - \phi_j) - i\omega_0\, \delta t_j\, \bar{\phi}_j\phi_{j-1} - \bar{\phi}_1(\phi_1 - \chi_0\phi_{2N}) \ , \tag{19}$$

where we have defined $\delta t_j = \Delta t$ for $j \le N$, $\delta t_{N+1} = 0$, and $\delta t_j = -\Delta t$ for $j > N+1$. We obtain

$$i\mathcal{S} = i\sum_{j=2}^{2N} \delta t_j \left[i\bar{\phi}_j \frac{(\phi_j - \phi_{j-1})}{\delta t_j} - \omega_0\, \bar{\phi}_j\phi_{j-1}\right] - \bar{\phi}_1(\phi_1 - \chi_0\phi_{2N}) \ . \tag{20}$$

This form would suggest an integral representation on the Keldysh contour ($\delta t_j < 0$ on the lower contour), if not the last term, which does not fit into it. We still use the integral representation, keeping in mind the last term:

$$i\mathcal{S} = i\int_K dt\, \bar{\phi}\, (i\partial_t - \omega_0)\, \phi \ . \tag{21}$$

II. INVERTED MATRIX iG^{-1}, CORRELATION FUNCTIONS

As before we do not really intend to invert the matrix (18). Rather we get it from the correlation functions (as above, with the same "cheating"). The proper way to do so is to introduce sources:

$$Z[\bar{J}, J] = (1 - \chi_0) \int \prod_{k=1}^{2N} d[\bar{\phi}_k, \phi_k] \, \exp\left[i \sum_{n,m} \bar{\phi}_m G^{-1}_{mn} \phi_n + \sum_k \bar{J}_k \phi_k + \sum_k J_k \bar{\phi}_k \right] . \tag{22}$$

The rules of Gaussian integration give

$$Z[\bar{J}, J] = (1 - \chi_0) \left[\mathrm{Det}(-\mathrm{i}G^{-1}) \right]^{-1} \exp\left[i \sum_{n,m} \bar{J}_m G_{mn} J_n \right] . \tag{23}$$

These rules follow from the Gaussian integration of a 1×1 matrix

$$\int d[\bar{\phi}, \phi] \exp\left[i\lambda\bar{\phi}\phi + J\bar{\phi} + \bar{J}\phi \right] = \frac{1}{(-i\lambda)} \exp\left[i\frac{\bar{J}J}{\lambda} \right] , \tag{24}$$

and the fact that matrix G^{-1} (with non-zero determinant) can be diagonalised by a rotation. Since $Z[\bar{J} = 0, J = 0] = 1$, we conclude that $(1 - \chi_0)\ [\mathrm{Det}(-\mathrm{i}G^{-1})]^{-1} = 1$ and

$$Z[\bar{J}, J] = \exp\left[i \sum_{n,m} \bar{J}_m G_{mn} J_n \right] . \tag{25}$$

From here we obtain

$$\langle \phi_m \bar{\phi}_n \rangle = \frac{\partial^2 Z}{\partial J_n \partial \bar{J}_m} \Big|_{J=0} = iG_{mn} . \tag{26}$$

Finally, looking closely at the construction procedure of the path integral one can see that

$$iG(t_n, t_m) \equiv \langle \phi_n \bar{\phi}_m \rangle = \langle T_K b(t_n) b^\dagger(t_m) \rangle . \tag{27}$$

For example, for the ++ component (both t_1 and t_2 belong to the upper Keldysh contour) this follows from

$$iG_{++}(t_1, t_2) = \begin{cases} \mathrm{Tr}\left[b e^{-(i/\hbar)H(t_1 - t_2)} b^\dagger e^{-(i/\hbar)Ht_2} \rho_0 \, e^{(i/\hbar)Ht_1} \right] & \text{if } t_1 > t_2 \\ \mathrm{Tr}\left[b^\dagger e^{-(i/\hbar)H(t_2 - t_1)} b e^{-(i/\hbar)Ht_1} \rho_0 \, e^{(i/\hbar)Ht_2} \right] & \text{if } t_2 > t_1 \end{cases} . \tag{28}$$

The calculation now is very similar to the one performed already for an oscillator in the previous lectures.

$$iG^> = iG_{-+} = \langle \phi_-(t_1) \bar{\phi}_+(t_2) \rangle = \langle b e^{-i\omega_0 t_1} b^\dagger e^{i\omega_0 t_2} \rangle = (n_B(\omega_0) + 1) e^{-i\omega_0(t_1 - t_2)} , \tag{29}$$

$$iG^< = iG_{+-} = \langle \phi_+(t_1) \bar{\phi}_-(t_2) \rangle = \langle b^\dagger e^{i\omega_0 t_2} b e^{i\omega_0 t_1} \rangle = n_B(\omega_0) e^{-i\omega_0(t_1 - t_2)} , \tag{30}$$

$$iG^c = iG_{++} = \langle \phi_+(t_1)\bar{\phi}_+(t_2)\rangle \; = \; \theta(t_1 - t_2) iG^>(t_1 - t_2) + \theta(t_2 - t_1) iG^<(t_1 - t_2) \; , \tag{31}$$

$$iG^{ac} = iG_{--} = \langle \phi_-(t_1)\bar{\phi}_-(t_2)\rangle \; = \; \theta(t_1 - t_2) iG^<(t_1 - t_2) + \theta(t_2 - t_1) iG^>(t_1 - t_2) \; . \tag{32}$$

Here $n_B(\omega_0) = (e^{\beta(\hbar\omega_0 - \mu)} - 1)^{-1}$. The Keldysh rotation to the quantum and classical components gives

$$G^R = \frac{1}{\omega - \omega_0 + i\delta} \; , \quad G^A = \frac{1}{\omega - \omega_0 - i\delta} \; , \quad G^K = (2n_B + 1)(G^R - G^A) \; . \tag{33}$$

If we want to use the integral representation (cf. Eqs. (20,21)) we have to rewrite Eq. (80) as

$$Z[\bar{J}, J] = (1 - \chi_0) \int \prod_{k=1}^{2N} d[\bar{\phi}_k, \phi_k] \, \exp\left[i\mathcal{S} + \sum_k \delta t_k \frac{\bar{J}_k}{\delta t_k} \phi_k + \sum_k \delta t_k \frac{J_k}{\delta t_k} \bar{\phi}_k \right] \; . \tag{34}$$

Thus we are forced to introduce the source "density" $J(t_k) = J_k/\delta t_k$. Note that it changes sign on the lower Keldysh contour.

III. BOSONIC BATH, INFLUENCE FUNCTIONAL FOR LINEAR COUPLING

What was done above for a single oscillator is directly generalised to the case of a bosonic bath, described by an arbitrary quadratic Hamiltonian

$$H_B = \hbar \sum_{k,p} b_k^\dagger h_{kp} b_p \; , \tag{35}$$

where h_{kp} is a Hermitian matrix (having dimension of frequency). We have a complex variable (operator) Q representing the system, such that the system-bath coupling reads $H_I = Q\bar{X} + \bar{Q}X$, where $X \equiv \sum_k \lambda_k b_k$. Then, repeating the calculations above we obtain for the influence functional

$$F[\bar{Q}, Q] = \exp\left[\frac{i}{\hbar^2} \int_K dt_1 dt_2 \, \bar{Q}(t_1) G_X(t_1, t_2) Q(t_2) \right] \; , \tag{36}$$

where

$$G_X(t_1, t_2) = -i \langle T_K X(t_1) \bar{X}(t_2) \rangle = \sum_{kp} \lambda_k \bar{\lambda}_p G_{kp}(t_1, t_2) \; . \tag{37}$$

Here, e.g., the retarded component of the matrix G_{kp} can be obtained as

$$G^R(\omega) = (\omega - \hat{h} + i\delta)^{-1} \ . \tag{38}$$

Here $\hat{h}$ is the Hermitian matrix from the Hamiltonian (35).

IV. BOSONIC BATH, INFLUENCE FUNCTIONAL FOR QUADRATIC COUPLING

Our formalism allows also for treating the case in which the coupling between the real (Hermitian) variable of the system V and the bath reads

$$H_I = \hbar V \sum_{kp} b_k^\dagger a_{kp} b_p \ . \tag{39}$$

Here $\hat{a}$ is Hermitian. The influence functional now reads

$$F[V] = \frac{1}{\mathrm{Tr}\left[e^{-\beta(H_B - \mu N)}\right]} \left[\mathrm{Det}(-\mathrm{i}\mathrm{G}^{-1})\right]^{-1} \ , \tag{40}$$

where "symbolically" $G^{-1} = (i\partial t - \hat{h} - V\hat{a}) = G_0^{-1} - V\hat{a}$.

We now try to understand this in a less symbolic way. The simplest way is to work with Keldysh rotated Green's functions. The function G_0 is the equilibrium one corresponding to the initial conditions. That is

$$\hat{G}_0^{-1}(\omega) = \begin{pmatrix} 0 & G_{00}^{-1} - \Sigma^A \\ G_{00}^{-1} - \Sigma^R & -\Sigma^K \end{pmatrix} \ , \tag{41}$$

where $G_{00}^{-1} = \omega - \hat{h}$, $\Sigma^R = -\Sigma^A = -i\delta$ (no sign(ω) needed as we work with b and $b^\dagger$ and not with real combinations $x \propto b + b^\dagger$). Finally (and crucially) $\Sigma^K(\omega) = -2i\delta \coth\left[\frac{\hbar\omega - \mu}{2k_B T}\right]$. Before the Keldysh rotation the perturbation term was looking like

$$G^{-1} = G_0^{-1} - \hat{a}\delta(t_1 - t_2) \begin{pmatrix} V_+(t_1) & 0 \\ 0 & -V_-(t_1) \end{pmatrix} \tag{42}$$

Upon rotation we obtain

$$\begin{aligned} \hat{G}^{-1} = LG^{-1}L \ &= \hat{G}_0^{-1} - \frac{1}{2}\hat{a}\delta(t_1 - t_2) \begin{pmatrix} V_+(t_1) - V_-(t_1) & V_+(t_1) + V_-(t_1) \\ V_+(t_1) + V_-(t_1) & V_+(t_1) - V_-(t_1) \end{pmatrix} \\ &= \hat{G}_0^{-1} - \frac{1}{\sqrt{2}}\hat{a}\delta(t_1 - t_2)\left[V_q(t_1)\tau_0 + V_c(t_1)\tau_x\right] \ . \end{aligned} \tag{43}$$

Introducing $\delta\hat{V} \equiv \frac{1}{\sqrt{2}}\,\hat{a}\delta(t_1 - t_2)\,[V_q(t_1)\tau_0 + V_c(t_1)\tau_x]$ we obtain

$$\begin{aligned}
F[V] &= \frac{1}{\mathrm{Tr}\left[e^{-\beta(H_B-\mu N)}\right]} e^{-\ln\left[\mathrm{Det}(-\mathrm{i}\mathrm{G}^{-1})\right]} \\
&= \frac{1}{\mathrm{Tr}\left[e^{-\beta(H_B-\mu N)}\right]} e^{-\mathrm{Tr}\ln\left[-iG^{-1}\right]} \\
&= \frac{1}{\mathrm{Tr}\left[e^{-\beta(H_B-\mu N)}\right]} e^{-\mathrm{Tr}\ln\left[-i\left(\hat{G}_0^{-1}-\delta V\right)\right]} \\
&= \frac{1}{\mathrm{Tr}\left[e^{-\beta(H_B-\mu N)}\right]} e^{-\mathrm{Tr}\ln\left[-i\hat{G}_0^{-1}\left(\hat{1}-\hat{G}_0\delta V\right)\right]} \\
&= \frac{1}{\mathrm{Tr}\left[e^{-\beta(H_B-\mu N)}\right]} e^{-\mathrm{Tr}\ln\left[-i\hat{G}_0^{-1}\right]-\mathrm{Tr}\ln\left(\hat{1}-\hat{G}_0\delta V\right)} \\
&= e^{-\mathrm{Tr}\ln\left(\hat{1}-\hat{G}_0\delta V\right)}\ .
\end{aligned} \tag{44}$$

V. PATH INTEGRAL FOR FERMIONS

In analogy to the case of bosons we consider a fermionic version of the harmonic oscillator described by the Hamiltonian

$$H = \epsilon_0\, c^\dagger c\ . \tag{45}$$

The operators c and $c^\dagger$ satisfy the usual fermionic commutation relations.

A. Grassmann numbers

In order to introduce the coherent states one needs Grassmann numbers. We will denote these by ψ, ψ', ψ_1, ψ_2 etc. The main properties of the Grassmann numbers are as follows:

$$\psi\psi' = -\psi'\psi \quad , \quad \psi^2 = 0\ . \tag{46}$$

One may introduce functions (polynomials) involving the Grassmann variables. For one Grassmann argument all the options are exhausted by

$$f(\psi) = f_0 + f_1\psi\ . \tag{47}$$

For two

$$f(\psi,\psi') = f_{00} + f_{10}\psi + f_{01}\psi' + f_{11}\,\psi\psi'\ . \tag{48}$$

Here the coefficients f are the complex numbers. This allows to differentiate, like $\partial f(\psi)/\partial\psi = f_1$. For a polynomial with two Grassmann numbers we get

$$\frac{\partial}{\partial\psi}\frac{\partial}{\partial\psi'}\,f(\psi,\psi') = \frac{\partial}{\partial\psi}(f_{01} - f_{11}\psi) = -f_{11}\ . \tag{49}$$

Alternatively

$$\frac{\partial}{\partial\psi'}\frac{\partial}{\partial\psi}f(\psi,\psi') = \frac{\partial}{\partial\psi'}(f_{10}+f_{11}\psi') = f_{11} \ . \tag{50}$$

Thus derivative anticommute. We also agree that the Gassmann numbers anti-commute with the Fermi operators: $\psi c = -c\psi$.

It might be clearer if one uses the language of wedge product. Then the Grassmann numbers (generators of the algebra) ψ_i are equivalent to basis vectors (1-forms) $\mathbf{e}_i$. Their product is the wedge product: $\psi_1\psi_2 \to \mathbf{e}_1 \wedge \mathbf{e}_2$. The whole thing then becomes the exterior algebra (Grassmann algebra) of the vector space spanned by $\mathbf{e}_i$ (over the field of complex numbers) .

One also introduces formally integration

$$\int d\psi\, 1 = 0 \quad , \quad \int d\psi\, \psi = 1 \ . \tag{51}$$

The integration is chosen to coincide with differentiation rather to be opposite to it.

B. Coherent states

We look for a state which would be an eigenstate of the annihilation operator. It cannot be a regular superposition of $|0\rangle$ and $|1\rangle$. Indeed $c(x\,|0\rangle + y\,|1\rangle) = y\,|0\rangle$, where x and y are complex numbers. It works with the Grassmann numbers. Namely, we consider a state $|\psi\rangle \equiv |0\rangle - \psi\,|1\rangle = (1-\psi c^\dagger)\,|0\rangle = \exp\left[-\psi c^\dagger\right]\,|0\rangle$. We obtain

$$c\,|\psi\rangle = c(|0\rangle - \psi\,|1\rangle) = \psi\,|0\rangle = \psi\,|\psi\rangle \ . \tag{52}$$

Since we have constructed the ket state $|\psi\rangle$ with the help of the Grassmann number, it is not clear how to construct the corresponding bra state $\langle\psi|$. The solution is to choose a different Grassmann number $\bar\psi$ and use it as complex conjugate to ψ, so that $\langle\psi| = \langle 0| - \langle 1|\,\bar\psi$. Then

$$\langle\psi|\,c^\dagger = (\langle 0| - \langle 1|\,\bar\psi)\,c^\dagger = \langle 1|\,c^\dagger\bar\psi = \langle 0|\,\bar\psi = \langle\psi|\,\bar\psi \ . \tag{53}$$

The choice of $\bar\psi$ is arbitrary. At the end all the Grassmann variables will be divided into pairs $(\psi_i, \bar\psi_i)$.

For the matrix element we obtain

$$\langle\psi_1|\psi_2\rangle = (\langle 0| - \langle 1|\,\bar\psi_1)(|0\rangle - \psi_2\,|1\rangle) = 1 + \bar\psi_1\psi_2 = e^{\bar\psi_1\psi_2} \ . \tag{54}$$

Resolution of unity

$$\hat{1} = \int d\bar{\psi} \int d\psi \, e^{-\bar{\psi}\psi} \, |\psi\rangle \, \langle\psi| \ . \tag{55}$$

It is easy to prove

$$e^{-\bar{\psi}\psi} \, |\psi\rangle \, \langle\psi| = (1 - \bar{\psi}\psi)(|0\rangle - \psi \, |1\rangle)(\langle 0| - \langle 1| \, \bar{\psi}) = \psi\bar{\psi} \, [|0\rangle \, \langle 0| + |1\rangle \, \langle 1|] + \dots \tag{56}$$

Here ... stand for terms of lower order in Grassmann variables, which vanish upon integration. To prove we use $\int d\bar{\psi} \int d\psi \, \psi\bar{\psi} = 1$.

Further, for a normal-ordered operator we obtain

$$\langle\psi_1| \, H(c^\dagger, c) \, |\psi_2\rangle = H(\bar{\psi}_1, \psi_2) \, \langle\psi_1|\psi_2\rangle = H(\bar{\psi}_1, \psi_2) \, e^{\bar{\psi}_1\psi_2} \tag{57}$$

Calculating trace:

$$\begin{aligned} \mathrm{Tr}[O] &= \sum_{n=0,1} \langle n| \, O \, |n\rangle = \sum_{n=0,1} \int d\bar{\psi} \int d\psi \, e^{-\bar{\psi}\psi} \, \langle n|\psi\rangle \, \langle\psi| \, O \, |n\rangle \\ &= \sum_{n=0,1} \int d\bar{\psi} \int d\psi \, e^{-\bar{\psi}\psi} \, \langle\psi| \, O \, |n\rangle \, \langle n| - \psi\rangle \\ &= \int d\bar{\psi} \int d\psi \, e^{-\bar{\psi}\psi} \, \langle\psi| \, O \, |{-\psi}\rangle \end{aligned} \tag{58}$$

Here $|{-\psi}\rangle = |0\rangle + \psi \, |1\rangle$. The sign of ψ has changed because of the changing the order of $|\psi\rangle$ and $\langle\psi|$.

Gaussian integration

$$\int d\bar{\psi} \int d\psi \, e^{-a\bar{\psi}\psi} = \int d\bar{\psi} \int d\psi \, (1 - a\bar{\psi}\psi) = a \ . \tag{59}$$

Let χ and $\bar{\chi}$ be another pair of Grassmann variables

$$\begin{aligned} Z[\bar{\chi}, \chi] &= \int d\bar{\psi} \int d\psi \, e^{-a\bar{\psi}\psi + \bar{\psi}\chi + \bar{\chi}\psi} \\ &= \int d\bar{\psi} \int d\psi \, (1 - a\bar{\psi}\psi + \bar{\psi}\chi + \bar{\chi}\psi + \bar{\psi}\chi\bar{\chi}\psi) \\ &= a - \chi\bar{\chi} = a \left(1 + \frac{\bar{\chi}\chi}{a}\right) = a \, e^{a^{-1}\bar{\chi}\chi} \ . \end{aligned} \tag{60}$$

For matrices we obtain

$$\int \prod_j^N d\bar{\psi}_j d\psi_j \, e^{-\sum_{nm}^N \bar{\psi}_n A_{nm} \psi_m} = \det A \ . \tag{61}$$

Indeed, only one term out of the expansion of the exponent survives. This is the term with exactly $2N$ Grassmann variables, equal to the number of integrals:

$$\int \prod_j^N d\bar{\psi}_j d\psi_j \, e^{-\sum_{nm}^N \bar{\psi}_n A_{nm} \psi_m} = \int \prod_j^N d\bar{\psi}_j d\psi_j \, \frac{1}{N!} \left(-\sum_{nm}^N \bar{\psi}_n A_{nm} \psi_m\right)^N \ . \tag{62}$$

In this product only the terms survive in which every Grassmann variable appears once. Thus we obtain a product of all A_{nm} with different pairs of indexes. The combinatorics gives exactly the determinant. Next, it turns our that one can shift the integration variables:

$$\begin{aligned} Z[\bar{\chi},\chi] &= \int \prod_j^N d\bar{\psi}_j d\psi_j \, \exp\left[-\sum_{nm}^N \bar{\psi}_n A_{nm}\psi_m + \sum_n \left(\bar{\psi}_n\chi_n + \bar{\chi}_n\psi_n\right)\right] \\ &= \int \prod_j^N d\bar{\psi}_j d\psi_j \, \exp\left[-(\bar{\psi} - \bar{\chi}A^{-1})A(\psi - A^{-1}\chi) + \bar{\chi}A^{-1}\chi\right] \\ &= \text{Det}[A] \, \exp\left[\bar{\chi}A^{-1}\chi\right] \; . \end{aligned} \tag{63}$$

C. Path integral for a single Fermionic level

In analogy to the case of bosons we consider a fermionic version of the harmonic oscillator described by the Hamiltonian

$$H = \epsilon_0 \, c^\dagger c \; . \tag{64}$$

The operators c and $c^\dagger$ satisfy the usual fermionic commutation relations.

We prepare the system in a thermal equilibrium state

$$\rho_0 = \frac{e^{-\beta(H-\mu N)}}{\text{Tr}\left[e^{-\beta(H-\mu N)}\right]} = \frac{1}{\text{Tr}\left[e^{-\beta(H-\mu N)}\right]} \, e^{-\beta(\epsilon_0-\mu)c^\dagger c} \; . \tag{65}$$

For the denominator we obtain

$$\text{Tr}\left[e^{-\beta(H-\mu N)}\right] = \sum_{n=0}^{1} e^{-\beta(\epsilon_0-\mu)n} = 1 + e^{-\beta(\epsilon_0-\mu)} \; . \tag{66}$$

Analogously to the bosonic case we study

$$1 = \text{Tr}\left[e^{-(i/\hbar)Ht} \, \rho_0 \, e^{(i/\hbar)Ht}\right] \; . \tag{67}$$

We try to rewrite this with the help of the path integral. We use again the Keldysh contour as shown in Fig. 1 and obtain

$$1 = \text{Tr}\left[\prod_{n=1}^{N-1} e^{-(i/\hbar)(t_{n+1}-t_n)H} \, \rho_0 \prod_{n=N+1}^{2N-1} e^{(i/\hbar)(t_{n+1}-t_n)H}\right] \; . \tag{68}$$

Now we insert resolutions of unity (55) at each t_n and obtain

$$\begin{aligned} 1 = &\int \prod_{n=1}^{2N} d[\bar{\psi}_n,\psi_n] \; \text{Tr}\,[\\ &|\psi_N\rangle \ldots \langle\psi_2| \, e^{-(i/\hbar)H\Delta t} \, |\psi_1\rangle \, e^{-\bar{\psi}_1\psi_1} \, \langle\psi_1| \, \rho_o \, |\psi_{2N}\rangle \, e^{-\bar{\psi}_{2N}\psi_{2N}} \, \langle\psi_{2N}| \, e^{(i/\hbar)H\Delta t} \, |\psi_{2N-1}\rangle \ldots . \langle\psi_{N+1}|\big] \; . \end{aligned} \tag{69}$$

Using (58) we obtain

$$1 = \int d\bar{\chi} d\chi e^{-\bar{\chi}\chi} \int \prod_{n=1}^{2N} d[\bar{\psi}_n, \psi_n]$$
$$\langle\chi|\psi_N\rangle \ldots \langle\psi_2| e^{-(i/\hbar)H\Delta t} |\psi_1\rangle e^{-\bar{\psi}_1\psi_1} \langle\psi_1| \rho_o |\psi_{2N}\rangle e^{-\bar{\psi}_{2N}\psi_{2N}} \langle\psi_{2N}| e^{(i/\hbar)H\Delta t} |\psi_{2N-1}\rangle \ldots . \langle\psi_{N+1}| - \chi\rangle \ . \tag{70}$$

The minus can be shifted inside

$$1 = \int d\bar{\chi} d\chi e^{-\bar{\chi}\chi} \int \prod_{n=1}^{2N} d[\bar{\psi}_n, \psi_n]$$
$$\langle\chi|\psi_N\rangle \ldots \langle\psi_2| e^{-(i/\hbar)H\Delta t} |\psi_1\rangle e^{-\bar{\psi}_1\psi_1} \langle\psi_1| \rho_o |-\psi_{2N}\rangle e^{-\bar{\psi}_{2N}\psi_{2N}} \langle\psi_{2N}| e^{(i/\hbar)H\Delta t} |\psi_{2N-1}\rangle \ldots . \langle\psi_{N+1}|\chi\rangle \ .$$
$$= \int \prod_{n=1}^{2N} d[\bar{\psi}_n, \psi_n]$$
$$\langle\psi_{N+1}|\psi_N\rangle \ldots \langle\psi_2| e^{-(i/\hbar)H\Delta t} |\psi_1\rangle e^{-\bar{\psi}_1\psi_1} \langle\psi_1| \rho_o |-\psi_{2N}\rangle e^{-\bar{\psi}_{2N}\psi_{2N}} \langle\psi_{2N}| e^{(i/\hbar)H\Delta t} |\psi_{2N-1}\rangle \ldots . |\psi_{N+1}\rangle \ . \tag{71}$$

For the matrix element of the density matrix we get

$$\langle\psi_1| \rho_o |-\psi_{2N}\rangle \ = \ \frac{1}{1 + e^{-\beta(\epsilon_0-\mu)}} \langle\psi_1| e^{-\beta(\epsilon_0-\mu)c^\dagger c} |-\psi_{2N}\rangle \tag{72}$$

The exponent is not normal ordered, so we cannot just substitute c by ψ. Instead we notice that $\hat{n} = c^\dagger c$ and $n^k = n$. Thus $e^{-an} = 1 - n + ne^{-a}$. Thus we obtain

$$\langle\psi_1| \rho_o |-\psi_{2N}\rangle = \frac{1}{1+\chi_0} e^{-\chi_0\bar{\psi}_1\psi_{2N}} \ , \tag{73}$$

where $\chi_0 = e^{-\beta(\epsilon_0-\mu)}$.

All in all we obtain

$$1 = (1+\chi_0)^{-1} \int \prod_{k=1}^{2N} d[\bar{\psi}_k, \psi_k] \ \exp\left[i\mathcal{S}\right] \ , \tag{74}$$

where

$$i\mathcal{S} = \sum_{n,m} \bar{\psi}_m (iG^{-1})_{mn} \psi_n \tag{75}$$

and the matrix iG^{-1} looks like

$$iG^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & -\boldsymbol{\chi_0} \\ h_- & -1 & 0 & 0 & 0 & 0 \\ 0 & h_- & -1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 0 \\ 0 & 0 & 0 & h_+ & -1 & 0 \\ 0 & 0 & 0 & 0 & h_+ & -1 \end{pmatrix} \ . \tag{76}$$

Here $h_{\mp} = 1 \mp i\epsilon_0 \Delta t$ and we took the example of $N = 3$. We rewrite

$$i\mathcal{S} = \sum_{j=2}^{2N} \bar{\psi}_j(\psi_{j-1} - \psi_j) - i\epsilon_0\, \delta t_j\, \bar{\psi}_j \psi_{j-1} - \bar{\psi}_1(\psi_1 + \chi_0 \psi_{2N}) \ , \tag{77}$$

where we have defined $\delta t_j = \Delta t$ for $j \le N$, $\delta t_{N+1} = 0$, and $\delta t_j = -\Delta t$ for $j > N+1$. We obtain

$$i\mathcal{S} = i\sum_{j=2}^{2N} \delta t_j \left[i\bar{\psi}_j \frac{(\psi_j - \psi_{j-1})}{\delta t_j} - \epsilon_0\, \bar{\psi}_j \psi_{j-1} \right] - \bar{\psi}_1(\psi_1 + \chi_0 \psi_{2N}) \ . \tag{78}$$

Symbolically we can write

$$i\mathcal{S} = i \int_K dt\, \bar{\psi}\, (i\partial_t - \epsilon_0)\, \psi \ . \tag{79}$$

However we should remember the boundary terms which are not included in this integral representation.

D. Inverted matrix iG^{-1}, correlation functions

As before we do not really intend to invert the matrix (18). Rather we get it from the correlation functions (as above, with the same "cheating"). The proper way to do so is to introduce sources:

$$Z[\bar{J}, J] = (1 + \chi_0)^{-1} \int \prod_{k=1}^{2N} d[\bar{\psi}_k, \psi_k]\, \exp\left[i \sum_{n,m} \bar{\psi}_m G^{-1}_{mn} \psi_n + \sum_k \bar{J}_k \psi_k + \sum_k J_k \bar{\psi}_k \right] \ . \tag{80}$$

This time the sources J_k are Grassmann variables. The rules of Gaussian integration give

$$Z[\bar{J}, J] = (1 + \chi_0)^{-1} \left[\mathrm{Det}(-\mathrm{i}\mathrm{G}^{-1}) \right] \exp\left[i \sum_{n,m} \bar{J}_m G_{mn} J_n \right] \ . \tag{81}$$

Since $Z[\bar{J} = 0, J = 0] = 1$, we conclude that $(1 + \chi_0)^{-1}\ [\mathrm{Det}(-\mathrm{i}\mathrm{G}^{-1})] = 1$ and

$$Z[\bar{J}, J] = \exp\left[i \sum_{n,m} \bar{J}_m G_{mn} J_n \right] \ . \tag{82}$$

From here we obtain

$$\langle \psi_m \bar{\psi}_n \rangle = \frac{\partial^2 Z}{\partial J_n \partial \bar{J}_m}\Big|_{J=0} = iG_{mn} \ . \tag{83}$$

Finally, looking closely at the construction procedure of the path integral one can see that

$$iG(t_n, t_m) \equiv \langle \phi_n \bar{\phi}_m \rangle = \langle T_K c(t_n) c^\dagger(t_m) \rangle \ . \tag{84}$$

For example, for the ++ component (both t_1 and t_2 belong to the upper Keldysh contour) this follows from

$$iG_{++}(t_1,t_2) = \begin{cases} \mathrm{Tr}\left[ce^{-(i/\hbar)H(t_1-t_2)}c^\dagger e^{-(i/\hbar)Ht_2}\,\rho_0\,e^{(i/\hbar)Ht_1}\right] & \text{if } t_1 > t_2 \\ \mathrm{Tr}\left[c^\dagger e^{-(i/\hbar)H(t_2-t_1)}ce^{-(i/\hbar)Ht_1}\,\rho_0\,e^{(i/\hbar)Ht_2}\right] & \text{if } t_2 > t_1 \end{cases} . \quad (85)$$

The calculation now is very similar to the one performed already for an oscillator in the previous lectures.

$$iG^> = iG_{-+} = \langle T_K \psi_-(t_1)\bar{\psi}_+(t_2)\rangle = \langle ce^{-i\epsilon_0 t_1}c^\dagger e^{i\epsilon_0 t_2}\rangle = (1-n_F(\epsilon_0))e^{-i\epsilon_0(t_1-t_2)} , \quad (86)$$

$$iG^< = iG_{+-} = \langle T_K \psi_+(t_1)\bar{\psi}_-(t_2)\rangle = -\langle c^\dagger e^{i\epsilon_0 t_2}ce^{i\epsilon_0 t_1}\rangle = -n_F(\epsilon_0)e^{-i\epsilon_0(t_1-t_2)} , \quad (87)$$

$$iG^c = iG_{++} = \langle T_K \psi_+(t_1)\bar{\psi}_+(t_2)\rangle \;=\; \theta(t_1-t_2)iG^>(t_1-t_2) + \theta(t_2-t_1)iG^<(t_1-t_2) , \quad (88)$$

$$iG^{ac} = iG_{--} = \langle T_K \psi_-(t_1)\bar{\psi}_-(t_2)\rangle \;=\; \theta(t_1-t_2)iG^<(t_1-t_2) + \theta(t_2-t_1)iG^>(t_1-t_2) . \quad (89)$$

Here $n_F(\epsilon_0) = (e^{\beta(\hbar\epsilon_0-\mu)}+1)^{-1}$. The Keldysh rotation to the quantum and classical components gives

$$G^R = \frac{1}{\omega-\epsilon_0+i\delta} , \quad G^A = \frac{1}{\omega-\epsilon_0-i\delta} , \quad G^K = (1-2n_F)(G^R-G^A) . \quad (90)$$

If we want to use the integral representation (cf. Eqs. (20,21)) we have to rewrite Eq. (80) as

$$Z[\bar{J},J] = (1+\chi_0)^{-1}\int\prod_{k=1}^{2N} d[\bar{\psi}_k,\psi_k]\,\exp\left[i\mathcal{S} + \sum_k \delta t_k \frac{\bar{J}_k}{\delta t_k}\psi_k + \sum_k \delta t_k \frac{J_k}{\delta t_k}\bar{\psi}_k\right] . \quad (91)$$

Thus we are forced to introduce the source "density" $J(t_k) = J_k/\delta t_k$. Note that it changes sign on the lower Keldysh contour.

VI. FERMIONIC BATH. EFFECTIVE ACTION

The calculation now proceeds analogously to the bosonic case. There are two main differences. Consider again the bath Hamiltonain

$$H_B = \hbar \sum_{k,p} f_k^\dagger h_{kp} f_p \ , \tag{92}$$

where f_k are the fermionic annihilation operators. The coupling between the system and the bath is again given by

$$H_I = \hbar V \sum_{kp} f_k^\dagger a_{kp} f_p \ . \tag{93}$$

Then the influence functional reads

$$F[V] = e^{\mathrm{Tr}\ln\left(\hat{1} - \hat{G}_0 \delta V\right)} \ . \tag{94}$$

The sign in the exponent is different from the bosonic case (Gaussian integral gives the determinant $[\mathrm{Det}(-\mathrm{i}\mathrm{G}^{-1})]$ and not $1/\left[\mathrm{Det}(-\mathrm{i}\mathrm{G}^{-1})\right]$ like in the bosonic case. The retarded Green's function is again given by

$$G^R(\omega) = (\omega - \hat{h} + i\delta)^{-1} \ . \tag{95}$$

However, the fluctuation-dissipation now reads

$$G^K(\omega) = \tanh\left(\frac{\hbar\omega}{2k_B T}\right)(G^R - G^A) \ . \tag{96}$$

VII. AMBEGAOKAR-ECKERN-SCHÖN EFFECTIVE ACTION

Consider a tunnel junction between two metallic leads (cf. Fig 2) biased by the external current I. The capacitance of the junction is denoted by C. In addition to the electrons we have an extra degree of freedom, the electric potential on the left lead V, which is at the same time the voltage drop across the junction. As in the previous lectures we will use as the dynamical variable the phase drop Φ, such that $\dot{\Phi} = V$. The electronic part of the Hamiltonian reads $H_{el} = H_L + H_R + H_T$, where

$$H_L = \sum_{k,\sigma} \epsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} \ . \tag{97}$$

$$H_R = \sum_{p,\sigma} \epsilon_p d_{p,\sigma}^\dagger d_{p,\sigma} \ . \tag{98}$$

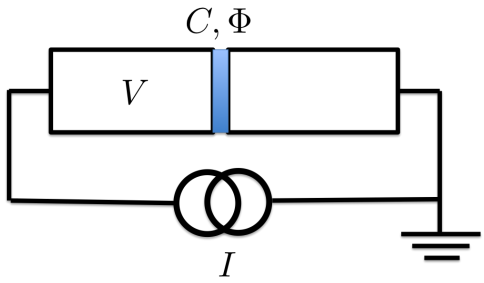

FIG. 2: Tunnel junction.

$$H_T = \sum_{k,p,\sigma} t_{k,p} c^\dagger_{k,\sigma} d_{p,\sigma} + h.c. \; . \tag{99}$$

The coupling energy between the electrons and the electric potential reads

$$H_I = eVN_L = eV \sum_{k,\sigma} c^\dagger_{k,\sigma} c_{k,\sigma} \; . \tag{100}$$

The full problem is the described by the following Lagrangian

$$\mathcal{L} = \frac{C\dot{\Phi}^2}{2} + I\Phi + \mathcal{L}_{el} + \mathcal{L}_I \; , \tag{101}$$

where $\mathcal{L}_{el}$ is the Lagrangian corresponding to the Hamiltonian H_{el}. Using Grassmann fields one can symbolically write

$$\mathcal{L}_{el} = \hbar i \, \bar{c} \, \partial_t c + \hbar i \, \bar{d} \, \partial_t d - H_{el}(\bar{c}, \bar{d}, c, d) \; . \tag{102}$$

Finally, the coupling Lagrangian reads

$$\mathcal{L}_I = -H_I(\dot{\Phi}, \bar{c}, c) = -e\dot{\Phi} \sum_{k,\sigma} \bar{c}_{k,\sigma} c_{k,\sigma} \; . \tag{103}$$

Our purpose is to integrate out the electrons and to obtain the effective action for Φ. We can easily do so since the coupling is quadratic in terms of the fermions.

Using the formalism developed above we obtain the influence functional of the following form

$$F[\Phi] = \mathcal{N} e^{\mathrm{Tr}\left[\ln(-iG^{-1})\right]} \; , \tag{104}$$

where

$$\hbar G^{-1} = \left(\begin{array}{c|c} \hbar G_L^{-1} - e\dot{\Phi} & -\hat{t}^\dagger \\ \hline -\hat{t} & \hbar G_R^{-1} \end{array} \right) \; . \tag{105}$$

This inverse Green's function is still written for the whole Keldysh contour. Thus, symbolically, $\hbar G_L^{-1} = \hbar i \partial_t - h_L$, and $\hbar G_R^{-1} = \hbar i \partial_t - h_R$, where $h_{L/R}$ are the first quantisation versions of $H_{L/R}$. As usual, we should keep in mind that this symbolic formulas are meaningful only if the initial state (density matrix) is properly taken into account in $G^{-1}_{L/R}$. This problematic does not concern the tunnelling parts as well as the coupling to $\dot{\Phi}$.

We now perform a gauge transformation

$$\hbar\tilde{G}^{-1} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{array}{c|c} \hbar G_L^{-1} - e\dot{\Phi} & -\hat{t}^\dagger \\ \hline -\hat{t} & \hbar G_R^{-1} \end{array}\right) \begin{pmatrix} U^\dagger & 0 \\ 0 & 1 \end{pmatrix} , \tag{106}$$

where $U = \exp\left[ie\Phi/\hbar\right]$. This essentially corresponds to the gauge transformation of the fermionic operators $c \to Uc$. The purpose is to eliminate the term $-e\dot{\Phi}$. The result reads

$$\hbar\tilde{G}^{-1} = \left(\begin{array}{c|c} \hbar G_L^{-1} & -U\hat{t}^\dagger \\ \hline -\hat{t}U^\dagger & \hbar G_R^{-1} \end{array}\right) , \tag{107}$$

Next we split

$$\tilde{G}^{-1} = \begin{pmatrix} G_L^{-1} & 0 \\ 0 & G_R^{-1} \end{pmatrix} + \frac{1}{\hbar} \begin{pmatrix} 0 & -U\hat{t}^\dagger \\ -\hat{t}U^\dagger & 0 \end{pmatrix} , \tag{108}$$

and consider the second part as a perturbation. This gives

$$F[\Phi] = \mathcal{N} e^{\mathrm{Tr}\left[\ln(-i\tilde{G}^{-1})\right]} = \exp\left(\mathrm{Tr}\ln\left[\hat{1} - \frac{1}{\hbar}\begin{pmatrix} G_L & 0 \\ 0 & G_R \end{pmatrix}\begin{pmatrix} 0 & U\hat{t}^\dagger \\ \hat{t}U^\dagger & 0 \end{pmatrix}\right]\right) . \tag{109}$$

(**Note**: For the normalisation pre-factor to drop we have to assume an adiabatic switching on of the tunnelling. Indeed our initial state of the reservoirs does not include tunnelling between them.) Expanding up to the second order in the tunnelling amplitudes we obtain the AES action:

$$(i/\hbar)\mathcal{S}_{AES} = \ln F[\Phi] = -\frac{1}{2\hbar^2}\mathrm{Tr}\left[\begin{pmatrix} G_L & 0 \\ 0 & G_R \end{pmatrix}\begin{pmatrix} 0 & U\hat{t}^\dagger \\ \hat{t}U^\dagger & 0 \end{pmatrix}\begin{pmatrix} G_L & 0 \\ 0 & G_R \end{pmatrix}\begin{pmatrix} 0 & U\hat{t}^\dagger \\ \hat{t}U^\dagger & 0 \end{pmatrix}\right] . \tag{110}$$

This gives

$$i\mathcal{S}_{AES} = -\frac{1}{\hbar}\,\mathrm{Tr}\left[G_L U\,\hat{t}^\dagger\,G_R\,\hat{t}\,U^\dagger\right] . \tag{111}$$

Eq. (111) is written without taking into account explicitly the Keldysh structure. We can thus define the kernel function

$$\alpha(t_1,t_2) \equiv \mathrm{Tr}\left[G_L(t_1,t_2)\hat{t}^\dagger\,G_R(t_2,t_1)\,\hat{t}\right] , \tag{112}$$

where t_1 and t_2 can be anywhere on the Keldysh contour. This allows us to rewrite (111) as

$$i\mathcal{S}_{AES} = -\frac{1}{\hbar} \int_K dt_1 dt_2 \, e^{-i(e/\hbar)\Phi(t_1)} \, \alpha(t_1, t_2) \, e^{i(e/\hbar)\Phi(t_2)} \ . \tag{113}$$

As we will see below, the kernel function is symmetric (if considered on the whole Keldysh contour), i.e., $\alpha(t_1, t_2) = \alpha(t_2, t_1)$. This allow us to obtain the following simple form of the AES action

$$i\mathcal{S}_{AES} = -\frac{1}{\hbar} \int_K dt_1 dt_2 \, \alpha(t_1, t_2) \, \cos\left[\frac{e(\Phi(t_1) - \Phi(t_2))}{\hbar}\right] \ . \tag{114}$$

A. The AES action in terms of quantum and classical fields

Now we introduce the Keldysh structure and take into account the initial density matrices of the reservoirs. Each block in (108) becomes a 2×2 Keldysh matrix. Each element of the Keldysh matrices is a matrix in the orbital indexes and in time. **Unlike in Kamenev's book, I use the same Keldysh rotation for fermions as it was used for bosons.** Thus we obtain

$$G_R^{-1} = \begin{pmatrix} 0 & G_{R0}^{-1} - \Sigma_R^A \\ G_{R0}^{-1} - \Sigma_R^R & -\Sigma_R^K \end{pmatrix} \ , \tag{115}$$

where $\hbar G_{R0}^{-1} = \hbar\omega - \epsilon_k$, $\Sigma_R^R = -\Sigma_R^A = -i\delta$, and $\Sigma_R^K = -2i\delta \tanh \frac{\hbar\omega - \mu_R}{2k_B T}$. Analogously

$$G_L^{-1} = \begin{pmatrix} 0 & G_{L0}^{-1} - \Sigma_L^A \\ G_{L0}^{-1} - \Sigma_L^R & -\Sigma_L^K \end{pmatrix} \ , \tag{116}$$

where $\hbar G_{L0}^{-1} = \hbar\omega - \epsilon_p$, $\Sigma_L^R = -\Sigma_L^A = -i\delta$, and $\Sigma_L^K = -2i\delta \tanh \frac{\epsilon - \mu_L}{2k_B T}$.

In analogy with (42) and (43) we obtain for, e.g., the upper right block of (108)

$$-U\hat{t}^\dagger \rightarrow -LU\hat{t}^\dagger L = -\frac{1}{\sqrt{2}} \left(U_q \tau_0 + U_c \tau_x\right) t^\dagger \ . \tag{117}$$

Here

$$U_c = (1/\sqrt{2}) \left(\exp[ie\Phi_+/\hbar] + \exp[ie\Phi_-/\hbar]\right) \ , \tag{118}$$

and

$$U_q = (1/\sqrt{2}) \left(\exp[ie\Phi_+/\hbar] - \exp[ie\Phi_-/\hbar]\right) \ . \tag{119}$$

We use again (111). After some algebra we obtain

$$i\mathcal{S}_{AES} = -\frac{1}{2\hbar} \int dt_1 dt_2 \begin{pmatrix} U_c^\dagger(t_1) & U_q^\dagger(t_1) \end{pmatrix} \begin{pmatrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{pmatrix}_{(t_1 - t_2)} \begin{pmatrix} U_c(t_2) \\ U_q(t_2) \end{pmatrix} \ , \tag{120}$$

where

$$\alpha^R(t_1-t_2) = \mathrm{Tr}\left[G^R_L(t_1-t_2)\,\hat{t}^\dagger\,G^K_R(t_2-t_1)\,\hat{t}\right] + \mathrm{Tr}\left[G^K_L(t_1-t_2)\,\hat{t}^\dagger\,G^A_R(t_2-t_1)\,\hat{t}\right] \ . \tag{121}$$

$$\alpha^A(t_1-t_2) = \mathrm{Tr}\left[G^A_L(t_1-t_2)\,\hat{t}^\dagger\,G^K_R(t_2-t_1)\,\hat{t}\right] + \mathrm{Tr}\left[G^K_L(t_1-t_2)\,\hat{t}^\dagger\,G^R_R(t_2-t_1)\,\hat{t}\right] \ . \tag{122}$$

$$\alpha^K(t_1-t_2) = \mathrm{Tr}\left[G^K_L(t_1-t_2)\,\hat{t}^\dagger\,G^K_R(t_2-t_1)\,\hat{t}\right] + \mathrm{Tr}\left[G^R_L(t_1-t_2)\,\hat{t}^\dagger\,G^A_R(t_2-t_1)\,\hat{t}\right] + \mathrm{Tr}\left[G^A_L(t_1-t_2)\,\hat{t}^\dagger\,G^R_R(t_2-t_1)\,\hat{t}\right] \ . \tag{123}$$

We start calculating the retarded part of the kernel $\alpha^R(t_1-t_2)$. We assume $\mu_L=\mu_R=\mu$ as the dynamical part of the voltage V is taken into account separately. We obtain

$$\begin{aligned}
\alpha^R(\omega) &= \int\frac{d\nu}{2\pi}\mathrm{Tr}\left[G^R_L(\nu+\omega)\,\hat{t}^\dagger\,G^K_R(\nu)\,\hat{t}\right]+\left[G^K_L(\nu+\omega)\,\hat{t}^\dagger\,G^A_R(\nu)\,\hat{t}\right] \\
&= -2\pi i\sum_{k,p,\sigma}|t_{kp}|^2\int\frac{d\nu}{2\pi}\left[\frac{\delta(\nu-\epsilon_p/\hbar)}{\nu+\omega-\epsilon_k/\hbar+i\delta}\ \tanh\left[\frac{\hbar\nu-\mu}{2k_BT}\right]\right. \\
&+ \left.\frac{\delta(\nu+\omega-\epsilon_k/\hbar)}{\nu-\epsilon_p/\hbar-i\delta}\ \tanh\left[\frac{\hbar(\nu+\omega)-\mu}{2k_BT}\right]\right] \\
&= -i\sum_{k,p,\sigma}|t_{kp}|^2\left[\frac{\tanh\left[\frac{\epsilon_p-\mu}{2k_BT}\right]-\tanh\left[\frac{\epsilon_k-\mu}{2k_BT}\right]}{\epsilon_p/\hbar-\epsilon_k/\hbar+\omega+i\delta}\right]
\end{aligned} \tag{124}$$

Let us calculate $\mathrm{Re}\,\alpha^R(\omega)$:

$$\mathrm{Re}\,\alpha^R(\omega) = -\pi\sum_{k,p,\sigma}|t_{kp}|^2\ \left[\tanh\left[\frac{\epsilon_p-\mu}{2k_BT}\right]-\tanh\left[\frac{\epsilon_k-\mu}{2k_BT}\right]\right]\delta(\epsilon_p/\hbar-\epsilon_k/\hbar+\omega) \ . \tag{125}$$

Assuming constant densities of states, constant tunnelling amplitudes and transforming to an integral we obtain

$$\begin{aligned}
\mathrm{Re}\,\alpha^R(\omega) &= -2\pi\hbar\rho_L\rho_R|t|^2\int d\epsilon\ \left[\tanh\left[\frac{\epsilon-\hbar\omega-\mu}{2k_BT}\right]-\tanh\left[\frac{\epsilon-\mu}{2k_BT}\right]\right] \\
&= 2\times 2\pi\rho_L\rho_R|t|^2\hbar^2\omega \ .
\end{aligned} \tag{126}$$

The separate factor 2 stands for spin. Taking into account our previous definition of dimensional conductance $g_T=2\times(2\pi)^2\rho_L\rho_R|t|^2$ (2 stands for spin), we obtain

$$\mathrm{Re}\,\alpha^R(\omega) = \frac{g_T}{2\pi}\hbar^2\omega \ . \tag{127}$$

B. Fluctuation-Dissipation relation

It is possible to show that the bosonic fluctuation-dissipation relation is satisfied for the kernel α. Namely

$$\alpha^K(\omega) = \coth\frac{\hbar\omega}{2k_BT}\left(\alpha^R(\omega) - \alpha^A(\omega)\right) . \tag{128}$$

It is also easy to show that $\alpha^A(\omega) = -[\alpha^R(\omega)]^*$, or equivalently, $\alpha^A(t) = -[\alpha^R(-t)]^*$. At the same time $\alpha^A(\omega) = \alpha^R(-\omega)$ and $\alpha^A(t) = \alpha^R(-t)$. Both relations are fulfilled, since

$$\mathrm{Re}\,\alpha^R(-\omega) = -\mathrm{Re}\,\alpha^R(\omega) \quad , \quad \mathrm{Im}\,\alpha^R(-\omega) = \mathrm{Im}\,\alpha^R(\omega) . \tag{129}$$

The relations $\alpha^A(\omega) = \alpha^R(-\omega)$ and $\alpha^A(t) = \alpha^R(-t)$ mean that $\alpha(t_1, t_2) = \alpha(t_2, t_1)$, where in the last equality the kernel function is considered on the whole Keldysh contour.

C. Semiclassical equation of motion

The full action describing the problem reads

$$i\mathcal{S} = i\int dt\,\left[C\dot{\Phi}_c(t)\dot{\Phi}_q(t) + \sqrt{2}I\,\Phi_q(t)\right] + i\mathcal{S}_{AES} . \tag{130}$$

For the dissipative part of the action we obtain $i\mathcal{S}_{AES} = i\mathcal{S}^R_{AES} + i\mathcal{S}^K_{AES}$, where

$$i\mathcal{S}^R_{AES} = -\frac{1}{2\hbar}\int dt_1 dt_2\,\left[U_q^\dagger(t_1)\alpha^R(t_1 - t_2)U_c(t_2) + U_q(t_1)\alpha^A(t_2 - t_1)U_c^\dagger(t_2)\right] , \tag{131}$$

and

$$i\mathcal{S}^K_{AES} = -\frac{1}{2\hbar}\int dt_1 dt_2\,\left[U_q^\dagger(t_1)\alpha^K(t_1 - t_2)U_q(t_2)\right] . \tag{132}$$

Taking into account $\alpha^A(t) = -[\alpha^R(-t)]^*$ we obtain

$$i\mathcal{S}^R_{AES} = -\frac{i}{\hbar}\int dt_1 dt_2\,\mathrm{Im}\left[U_q^\dagger(t_1)\alpha^R(t_1 - t_2)U_c(t_2)\right] , \tag{133}$$

To simplify the variation procedure we rewrite Eqs. (118,119) as

$$U_c = \sqrt{2}\,e^{i\phi_c/\sqrt{2}}\cos[\phi_q/\sqrt{2}] , \tag{134}$$

and

$$U_q = \sqrt{2}\,i\,e^{i\phi_c/\sqrt{2}}\sin[\phi_q/\sqrt{2}] . \tag{135}$$

Here, for brevity, we reintroduce the dimensionless phases $\phi \equiv e\Phi/\hbar = (1/2\pi)(\Phi/\Phi_0)$. Then, using $\alpha^A(t) = \alpha^R(-t)$ and $\alpha^K(t) = \alpha^K(-t)$ we can rewrite

$$i\mathcal{S}^R_{AES} \;=\; \frac{2}{\hbar}\int dt_1 dt_2\, \alpha^R(t_1,t_2)\, \sin\left[\frac{\phi_c(t_1)-\phi_c(t_2)}{\sqrt{2}}\right] \sin\frac{\phi_q(t_1)}{\sqrt{2}} \cos\frac{\phi_q(t_2)}{\sqrt{2}}\;. \tag{136}$$

$$i\mathcal{S}^K_{AES} = -\frac{1}{\hbar}\int dt_1 dt_2\, \alpha^K(t_1,t_2)\, \cos\left[\frac{\phi_c(t_1)-\phi_c(t_2)}{\sqrt{2}}\right] \sin\frac{\phi_q(t_1)}{\sqrt{2}} \sin\frac{\phi_q(t_2)}{\sqrt{2}}\;. \tag{137}$$

The last form is very convenient for variation procedure, which gives (only $\mathcal{S}^R$ contributes)

$$0 = -i\,C\,\ddot{\Phi}_c(t) + i\,I\,\sqrt{2} + \frac{2}{\hbar}\frac{e}{\sqrt{2}\hbar}\int dt_2\, \alpha^R(t-t_2)\, \sin\left[\frac{\phi_c(t_1)-\phi_c(t_2)}{\sqrt{2}}\right]\;. \tag{138}$$

Using (127) we replace $\alpha^R(t) \to i(\hbar^2 g_T/2\pi)\delta'(t)$. The result reads

$$0 = C\,\ddot{\Phi}_c(t) - I\,\sqrt{2} + \frac{\dot{\Phi}_c(t)}{R_T}\;, \tag{139}$$

where $(1/R_T) = g_T(e^2/2\pi\hbar)$. For the "proper" classical component $\tilde{\Phi}_c = (\Phi_+ + \Phi_-)/2 = \Phi_c/\sqrt{2}$ we, thus, obtain

$$C\frac{d^2\tilde{\Phi}_c}{dt^2} + \frac{1}{R_T}\frac{d\tilde{\Phi}_c}{dt} = I\;. \tag{140}$$

This is exactly the equation of motion of a current biased RC circuit as expected.

D. Langevin equation

We are now ready to explore the role of the second part of the effective dissipative action, i.e., $\mathcal{S}^K_{AES}$ (see Eq. (137)), which we rewrite as

$$\begin{aligned} i\mathcal{S}^K_{AES} \;=\; &-\frac{1}{\hbar}\int dt_1 dt_2\, \alpha^K(t_1,t_2)\, \cos\frac{\phi_c(t_1)}{\sqrt{2}} \cos\frac{\phi_c(t_2)}{\sqrt{2}} \sin\frac{\phi_q(t_1)}{\sqrt{2}} \sin\frac{\phi_q(t_2)}{\sqrt{2}} \\ &-\frac{1}{\hbar}\int dt_1 dt_2\, \alpha^K(t_1,t_2)\, \sin\frac{\phi_c(t_1)}{\sqrt{2}} \sin\frac{\phi_c(t_2)}{\sqrt{2}} \sin\frac{\phi_q(t_1)}{\sqrt{2}} \sin\frac{\phi_q(t_2)}{\sqrt{2}}\;. \end{aligned} \tag{141}$$

We introduce two Hubbard-Stratonovich fields $\xi_1(t)$ and $\xi_2(t)$ and obtain

$$\begin{aligned} \exp\left[(i/\hbar)\mathcal{S}^K_{AES}\right] \;=\; &\int D\xi_1\, \exp\left[(i/\hbar^2)\int dt\,\sqrt{2}\xi_1(t)\, \cos\frac{\phi_c(t)}{\sqrt{2}}\, \sin\frac{\phi_q(t)}{\sqrt{2}}\right] \\ &\times\; \exp\left[-\frac{1}{2\hbar^2}\int dt_1 dt_2\, \left[\xi_1(t_1)(\alpha^K)^{-1}(t_1,t_2)\xi_1(t_2)\right]\right] \\ &\times\; \int D\xi_2\, \exp\left[(i/\hbar^2)\int dt\,\sqrt{2}\xi_2(t)\, \sin\frac{\phi_c(t)}{\sqrt{2}}\, \sin\frac{\phi_q(t)}{\sqrt{2}}\right] \\ &\times\; \exp\left[-\frac{1}{2\hbar^2}\int dt_1 dt_2\, \left[\xi_2(t_1)(\alpha^K)^{-1}(t_1,t_2)\xi_2(t_2)\right]\right]\;. \end{aligned} \tag{142}$$

As a result we have an extra term in the action linear in Φ_q, which reads

$$i\mathcal{S}_\xi = (\sqrt{2}i/\hbar)\int dt \left(\xi_1(t)\cos\frac{\phi_c(t)}{\sqrt{2}}\,\sin\frac{\phi_q(t)}{\sqrt{2}} + \xi_2(t)\sin\frac{\phi_c(t)}{\sqrt{2}}\,\sin\frac{\phi_q(t)}{\sqrt{2}}\right) , \tag{143}$$

whereas the Langevin fields ξ_1 and ξ_2 are described by the correlation functions

$$\langle \xi_\alpha(t_1)\xi_\beta(t_2)\rangle = \hbar^2\alpha^K(t_1-t_2)\,\delta_{\alpha,\beta} . \tag{144}$$

We now vary the action with respect to Φ_q

$$i\mathcal{S} = i\int dt\,\left[C\dot{\Phi}_c(t)\dot{\Phi}_q(t) + \sqrt{2}I\,\Phi_q(t)\right] + i\mathcal{S}^R_{AES} + i\mathcal{S}_\xi \tag{145}$$

and obtain the quasi-classical equation of motion

$$C\,\ddot{\Phi}_c(t) + \frac{\dot{\Phi}_c(t)}{R_T} = I\,\sqrt{2} + \frac{e}{\hbar^2}\cos\left(\frac{\phi_c}{\sqrt{2}}\right)\,\xi_1 + \frac{e}{\hbar^2}\sin\left(\frac{\phi_c}{\sqrt{2}}\right)\,\xi_2 . \tag{146}$$

We divide by $\sqrt{2}$ (in order to get the equation for the proper classical component $\tilde{\Phi}_c$) and obtain

$$C\frac{d^2\tilde{\Phi}_c}{dt^2} + \frac{1}{R_T}\frac{d\tilde{\Phi}_c}{dt} = I + \delta I_1 + \delta I_2 , \tag{147}$$

where

$$\delta I_1 \equiv \frac{e}{\sqrt{2}\hbar^2}\cos\left(\frac{\phi_c}{\sqrt{2}}\right)\,\xi_1 = \frac{e}{\sqrt{2}\hbar^2}\cos\left(2\pi\,\frac{\tilde{\Phi}_c}{\Phi_0}\right) , \tag{148}$$

and

$$\delta I_2 \equiv \frac{e}{\sqrt{2}\hbar^2}\sin\left(\frac{\phi_c}{\sqrt{2}}\right)\,\xi_2 = \frac{e}{\sqrt{2}\hbar^2}\sin\left(2\pi\,\frac{\tilde{\Phi}_c}{\Phi_0}\right)\,\xi_2 . \tag{149}$$

At equilibrium, $I=0$, we can put $\tilde{\Phi}_c = const.$ into the $\cos(\ldots)$ and $\sin(\ldots)$ in the RHS of (148) and (149). Then we easily obtain

$$\langle \delta I_1\delta I_1\rangle_\omega + \langle \delta I_2\delta I_2\rangle_\omega = \frac{\hbar\omega}{R_T}\,\coth\left(\frac{\hbar\omega}{2K_BT}\right) . \tag{150}$$

Out of equilibrium, i.e., $I \neq 0$, we look for the solution of the form

$$\tilde{\Phi}_c = Vt + \delta\Phi , \tag{151}$$

where $V = R_T I$. Neglecting $\delta\Phi$ in the RHS of (147), i.e., in $\cos(\ldots)$ and $\sin(\ldots)$ in the RHS of (148) and (149), we obtain

$$\langle \delta I_1\delta I_1\rangle_\omega = \langle \delta I_2\delta I_2\rangle_\omega = \frac{1}{4}\left[\frac{\hbar\omega+eV}{R_T}\,\coth\left(\frac{\hbar\omega+eV}{2K_BT}\right) + \frac{\hbar\omega-eV}{R_T}\,\coth\left(\frac{\hbar\omega-eV}{2K_BT}\right)\right] . \tag{152}$$

(We have dropped the terms depending on $t_1 + t_2$). In particular, at $\omega = 0$ this gives

$$\langle \delta I_1 \delta I_1 \rangle_{\omega=0} = \langle \delta I_2 \delta I_2 \rangle_{\omega=0} = \frac{1}{2}\frac{eV}{R_T} \coth\left(\frac{eV}{2K_BT}\right) \ . \tag{153}$$

At $eV \gg k_BT$ we obtain the usual shot noise:

$$\langle \delta I_1 \delta I_1 \rangle_{\omega=0} + \langle \delta I_2 \delta I_2 \rangle_{\omega=0} \approx \frac{eV}{R_T} = eI \ . \tag{154}$$

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[1] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, 2011).