Selected topics in solid state physics 2

Part 6

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I. ANDERSON-HIGGS MECHANISM

Consider a superconductor in an electromagnetic field

$$H - \mu N = \sum_\sigma \int dr\, \Psi^\dagger_\sigma \left[\frac{1}{2m}\left(-i\vec{\nabla} - e\vec{A}\right)^2 + eV - \mu\right]\Psi_\sigma - g\int dr \Psi^\dagger_\uparrow \Psi^\dagger_\downarrow \Psi_\downarrow \Psi_\uparrow \ . \quad (1)$$

The action reads

$$i\mathcal{S} = i\sum_\sigma \int dtdr\, \bar{\Psi}_\sigma \left[i\partial_t - \frac{1}{2m}\left(-i\vec{\nabla} - e\vec{A}\right)^2 - eV + \mu\right]\Psi_\sigma + ig\int dtdr \bar{\Psi}_\uparrow \bar{\Psi}_\downarrow \Psi_\downarrow \Psi_\uparrow \ . \quad (2)$$

Hubbard-Stratonovich

$$\begin{aligned} i\mathcal{S} \ &= \ i\sum_\sigma \int dtdr\, \bar{\Psi}_\sigma \left[i\partial_t - \frac{1}{2m}\left(-i\vec{\nabla} - e\vec{A}\right)^2 - eV + \mu\right]\Psi_\sigma \\ &+ \ i\int dtdr \Delta \bar{\Psi}_\uparrow \bar{\Psi}_\downarrow + i\int dtdr \Delta^* \Psi_\downarrow \Psi_\uparrow - i\int dtdr \frac{|\Delta|^2}{g} \ . \end{aligned} \quad (3)$$

We have $\Delta = |\Delta| e^{-i\varphi}$. Gauge transformation $\Psi_\sigma \to \Psi_\sigma e^{-i\varphi/2}$, $eV \to eV - \dot{\varphi}/2$, $e\vec{A} \to e\vec{A} + \vec{\nabla}\varphi/2$. We obtain

$$\begin{aligned} i\mathcal{S} \ &= \ i\sum_\sigma \int dtdr\, \bar{\Psi}_\sigma \left[i\partial_t - \frac{1}{2m}\left(-i\vec{\nabla} - e\vec{v}\right)^2 - e\Phi + \mu\right]\Psi_\sigma \\ &+ \ i\int dtdr |\Delta| \bar{\Psi}_\uparrow \bar{\Psi}_\downarrow + i\int dtdr |\Delta| \Psi_\downarrow \Psi_\uparrow - i\int dtdr \frac{|\Delta|^2}{g} \ , \end{aligned} \quad (4)$$

where $\vec{v} \equiv \vec{A} + \frac{\nabla\varphi}{2e}$ and $\Phi = V - \frac{\dot{\phi}}{2e}$. One essentially gets a new (gauge-invariant) vector-potential $(V, \vec{A}) \to (\Phi, \vec{v})$. After integrating our the electrons and optimising Δ (BCS mean-field) and μ (canonical- grand-canonical) the effective electromagnetic action reads

$$\mathcal{S}_{Higgs} = \frac{1}{2}\int dtdr \left(c_1 \Phi^2 - c_2 (\vec{v})^2\right) \ , \quad (5)$$

where $c_1 \approx 2e^2 \nu_F$ and $c_2 \approx \frac{e^2 n_s}{mc^2}$. Here n_s is the "density of superconducting electrons". At $T = 0$ we have $n_s = n$. To this action one should add

$$\mathcal{S}_{em} = \int dtdr \frac{E^2 - B^2}{8\pi} \ . \quad (6)$$

Result: Meissner effect, plasmons. At low frequencies Josephson relation: $V = \frac{\hbar\dot{\varphi}}{2e}$.

II. AES ACTION FOR A JOSEPHSON JUNCTION

We consider again a tunnel junction (Fig. 1). This time both metals are superconducting. The mean-field BCS Hamiltonian of the left lead reads

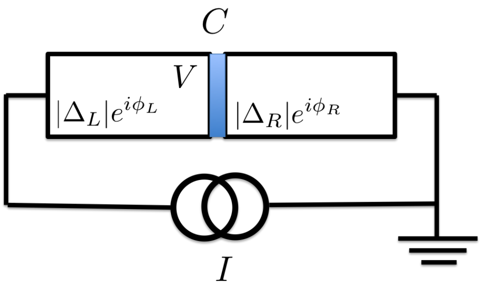

FIG. 1: Josephson junction.

$$H_L = \sum_{k,\sigma} \xi_k c^\dagger_{k,\sigma} c_{k,\sigma} - \sum_k \Delta^*_L c_{-k,\downarrow} c_{k,\uparrow} - \sum_k \Delta_L c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} + const. \tag{7}$$

Here $\Delta_L = |\Delta_L|e^{i\phi_L}$, where ϕ_L is the phase of the order parameter of the left lead. The usual trick leads to

$$H_L = \sum_k \xi_k c^\dagger_{k,\uparrow} c_{k,\uparrow} - \sum_k \xi_k c_{k,\downarrow} c^\dagger_{k,\downarrow} - \sum_k \Delta^*_L c_{-k,\downarrow} c_{k,\uparrow} - \sum_k \Delta_L c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} + const. \tag{8}$$

and

$$H_L = \sum_k \begin{pmatrix} c^\dagger_{k,\uparrow} & c_{-k,\downarrow} \end{pmatrix} \begin{pmatrix} \xi_k & -\Delta_L \\ -\Delta^*_L & -\xi_k \end{pmatrix} \begin{pmatrix} c_{k,\uparrow} \\ c^\dagger_{-k,\downarrow} \end{pmatrix} . \tag{9}$$

Analogously we write the Hamiltonian of the right lead:

$$H_L = \sum_k \begin{pmatrix} d^\dagger_{k,\uparrow} & d_{-k,\downarrow} \end{pmatrix} \begin{pmatrix} \xi_k & -\Delta_R \\ -\Delta^*_R & -\xi_k \end{pmatrix} \begin{pmatrix} d_{k,\uparrow} \\ d^\dagger_{-k,\downarrow} \end{pmatrix} . \tag{10}$$

Here $\Delta_R = |\Delta_R|e^{i\phi_R}$. In addition there is the tunnelling Hamiltonian:

$$H_T = \sum_{k,p,\sigma} t_{k,p} c^\dagger_{k,\sigma} d_{p,\sigma} + h.c. \; . \tag{11}$$

and the coupling energy between the electrons and the electric potential:

$$H_I = eVN_L = eV \sum_{k,\sigma} c^\dagger_{k,\sigma} c_{k,\sigma} \; . \tag{12}$$

Using the formalism developed above we obtain the influence functional of the following form

$$F[V,\phi_L,\phi_R] = \mathcal{N} e^{\mathrm{Tr}\left[\ln(-iG^{-1})\right]} \; , \tag{13}$$

where

$$\hbar G^{-1} = \left(\begin{array}{c|c} \hbar G^{-1}_L - eV\eta_z & -\hat{t}^\dagger \eta_z \\ \hline -\hat{t}\,\eta_z & \hbar G^{-1}_R \end{array}\right) . \tag{14}$$

Here η_z is the Pauli matrix in the Nambu space. (We use $\vec{\tau}$ for Keldysh, $\vec{\sigma}$ for spin, and $\vec{\eta}$ for Nambu.) This inverse Green's function is still written for the whole Keldysh contour. Thus, symbolically, $\hbar G_L^{-1} = \hbar i\partial_t - h_L$, and $\hbar G_R^{-1} = \hbar i\partial_t - h_R$, where $h_{L/R}$ are the first quantisation versions of $H_{L/R}$. We have

$$h_L = \begin{pmatrix} \xi_k & -\Delta_L \\ -\Delta_L^* & -\xi_k \end{pmatrix} \quad , \quad h_R = \begin{pmatrix} \xi_k & -\Delta_R \\ -\Delta_R^* & -\xi_k \end{pmatrix} . \tag{15}$$

We now perform a gauge transformation

$$\hbar\tilde{G}^{-1} = \begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} \left(\begin{array}{c|c} \hbar G_L^{-1} - eV\eta_z & -\hat{t}^\dagger\,\eta_z \\ \hline -\hat{t}\,\eta_z & \hbar G_R^{-1} \end{array}\right) \begin{pmatrix} U_L^\dagger & 0 \\ 0 & U_R^\dagger \end{pmatrix} , \tag{16}$$

where $U_L = \begin{pmatrix} e^{i\chi_L} & 0 \\ 0 & e^{-i\chi_L} \end{pmatrix} = e^{i\chi_L\eta_z}$ and $U_R = \begin{pmatrix} e^{i\chi_R} & 0 \\ 0 & e^{-i\chi_R} \end{pmatrix} = e^{i\chi_R\eta_z}$. The result reads

$$\hbar\tilde{G}^{-1} = \left(\begin{array}{c|c} \hbar\tilde{G}_L^{-1} - (eV - \hbar\dot{\chi}_L)\eta_z & -U_L\hat{t}^\dagger\,\eta_z U_R^\dagger \\ \hline -U_R\hat{t}\,\eta_z U_L^\dagger & \hbar\tilde{G}_R^{-1} + \hbar\dot{\chi}_R\eta_z \end{array}\right) , \tag{17}$$

where

$$\tilde{h}_L = \begin{pmatrix} \xi_k & -\Delta_L e^{2i\chi_L} \\ -\Delta_L^* e^{-2i\chi_L} & -\xi_k \end{pmatrix} \quad , \quad \tilde{h}_R = \begin{pmatrix} \xi_k & -\Delta_R e^{2i\chi_R} \\ -\Delta_R^* e^{-2i\chi_R} & -\xi_k \end{pmatrix} . \tag{18}$$

For the right lead we choose $2\chi_R = -\phi_R$. Then the order parameter of the left lead becomes real. However we generate a scalar potential $-\hbar\dot{\chi}_R/e$. For the left lead the similar choice $2\chi_L = -\phi_L$ makes the order parameter of the left lead real wheres if we choose $\hbar\dot{\chi}_L = eV$, the electric potential would vanish. Fortunately we have the Josephson relations between the phase of the order parameter and the scalar potential: $\hbar\dot{\phi}_L = -2eV_L = -2eV$ and $\hbar\dot{\phi}_R = -2eV_R = 0$. These relations are usually derived in the field theory of superconductivity and constitute the famous Higgs mechanism. More precisely it is shown that there is a term in the free energy proportional to $(\hbar\dot{\phi}_{L/R} + 2eV_{L/R})^2$. The coefficient in front of this term in the mass generated by the Higgs mechanism. Thus the time derivative of the phase is pinned to the electrostatic potential. In the tunnelling terms we obtain the phase factors

$$U_R\hat{t}\,\eta_z U_L^\dagger = \begin{pmatrix} \hat{t}e^{i\phi/2} & 0 \\ 0 & -\hat{t}e^{-i\phi/2} \end{pmatrix} , \tag{19}$$

where $\phi/2 \equiv \chi_R - \chi_L = (\phi_L - \phi_R)/2 = const. - (e/\hbar)\int dt' V(t')$. For the effective action we obtain

$$i\mathcal{S}_{AES} = -\frac{1}{\hbar}\,\mathrm{Tr}\left[G_L U_L\,\hat{t}^\dagger\,\eta_z\,U_R^\dagger\,G_R\,U_R\hat{t}\,\eta_z\,U_L^\dagger\right]\ . \tag{20}$$

The Gor'kov's Green's functions read:

$$\begin{aligned} G_L(t_1,t_2) &= \begin{pmatrix} g_L(t_1,t_2) & \tilde{f}_L(t_1,t_2) \\ f_L(t_1,t_2) & \tilde{g}_L(t_1,t_2) \end{pmatrix} \\ &= -i\begin{pmatrix} \langle T_K c_{k,\uparrow}(t_1) c^\dagger_{k,\uparrow}(t_2)\rangle & \langle T_K c^\dagger_{-k,\downarrow}(t_1) c^\dagger_{k,\uparrow}(t_2)\rangle \\ \langle T_K c_{k,\uparrow}(t_1) c_{-k,\downarrow}(t_2)\rangle & \langle T_K c^\dagger_{-k,\downarrow}(t_1) c_{-k,\downarrow}(t_2)\rangle \end{pmatrix} \end{aligned} \tag{21}$$

and similarly for G_R. We observe $\tilde{g}(t_1,t_2) = -g(t_2,t_1)$. We obtain

$$i\mathcal{S}_{AES} = -\frac{1}{\hbar}\int dt_1 dt_2 \mathrm{Tr}\left[\begin{pmatrix} g & \tilde{f} \\ f & \tilde{g}\end{pmatrix}_{(t_1,t_2)} \begin{pmatrix} \hat{t}^\dagger e^{-i\phi/2} & 0 \\ 0 & -\hat{t}^\dagger e^{i\phi/2}\end{pmatrix}_{t_2} \begin{pmatrix} g & \tilde{f} \\ f & \tilde{g}\end{pmatrix}_{(t_2,t_1)} \begin{pmatrix} \hat{t} e^{i\phi/2} & 0 \\ 0 & -\hat{t} e^{-i\phi/2}\end{pmatrix}_{t_1}\right]\ . \tag{22}$$

Further

$$\begin{aligned} i\mathcal{S}_{AES} = &- \frac{1}{\hbar}\int dt_1 dt_2 \mathrm{Tr}\left[g_{(t_1,t_2)}\,\hat{t}^\dagger\,g_{(t_2,t_1)}\,\hat{t}\right]\,e^{\frac{i(\phi(t_1)-\phi(t_2))}{2}} \\ &- \frac{1}{\hbar}\int dt_1 dt_2 \mathrm{Tr}\left[\tilde{g}_{(t_1,t_2)}\,\hat{t}^\dagger\,\tilde{g}_{(t_2,t_1)}\,\hat{t}\right]\,e^{\frac{i(\phi(t_2)-\phi(t_1))}{2}} \\ &+ \frac{1}{\hbar}\int dt_1 dt_2 \mathrm{Tr}\left[f_{(t_1,t_2)}\,\hat{t}^\dagger\,\tilde{f}_{(t_2,t_1)}\,\hat{t}\right]\,e^{-\frac{i(\phi(t_1)+\phi(t_2))}{2}} \\ &+ \frac{1}{\hbar}\int dt_1 dt_2 \mathrm{Tr}\left[\tilde{f}_{(t_1,t_2)}\,\hat{t}^\dagger\,f_{(t_2,t_1)}\,\hat{t}\right]\,e^{\frac{i(\phi(t_1)+\phi(t_2))}{2}}\ . \end{aligned} \tag{23}$$

Taking into account the symmetry properties, i.e., $\tilde{g}(t_1,t_2) = -g(t_2,t_1)$, we obtain

$$i\mathcal{S}_{AES} = -\frac{2}{\hbar}\int dt_1 dt_2 \left[\alpha(t_1,t_2)\,\cos\left[\frac{\phi(t_1)-\phi(t_2)}{2}\right] + \beta(t_1,t_2)\,\cos\left[\frac{\phi(t_1)+\phi(t_2)}{2}\right]\right]\ , \tag{24}$$

where

$$\alpha(t_1,t_2) = \mathrm{Tr}\left[g(t_1,t_2)\hat{t}^\dagger g(t_2,t_1)\hat{t}\right]\ . \tag{25}$$

and

$$\beta(t_1,t_2) = -\mathrm{Tr}\left[f(t_1,t_2)\hat{t}^\dagger \tilde{f}(t_2,t_1)\hat{t}\right]\ . \tag{26}$$

Next we explore the Keldysh structure of these kernel functions. For the Green's function of the (left) lead we obtain

$$G_L^R = \hbar\begin{pmatrix} \hbar\omega - \epsilon_k + i\delta & -|\Delta_L| \\ -|\Delta_L| & \hbar\omega + \epsilon_k + i\delta \end{pmatrix}^{-1} = \frac{\hbar\begin{pmatrix} \hbar\omega + \epsilon_k & |\Delta_L| \\ |\Delta_L| & \hbar\omega - \epsilon_k \end{pmatrix}}{\hbar^2(\omega+i\delta)^2 - (|\Delta_L|^2 + \epsilon_k^2)}\ . \tag{27}$$

The advanced component is given by $G_L^A(\omega) = G_L^R(\omega)^*$ and, as usual, $G_L^K = \tanh(\hbar\omega/2k_BT)(G_L^R - G_L^A)$. For the effective action we obtain

$$\begin{aligned} i\mathcal{S}_{AES} = & - \frac{1}{2\hbar}\int dt_1 dt_2 \left(\left[e^{-i\phi(t_1)/2}\right]_c \; \left[e^{-i\phi(t_1)/2}\right]_q \right) \begin{pmatrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{pmatrix}_{(t_1-t_2)} \begin{pmatrix} \left[e^{i\phi(t_2)/2}\right]_c \\ \left[e^{i\phi(t_2)/2}\right]_q \end{pmatrix} \\ & - \frac{1}{2\hbar}\int dt_1 dt_2 \left(\left[e^{i\phi(t_1)/2}\right]_c \; \left[e^{i\phi(t_1)/2}\right]_q \right) \begin{pmatrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{pmatrix}_{(t_1-t_2)} \begin{pmatrix} \left[e^{-i\phi(t_2)/2}\right]_c \\ \left[e^{-i\phi(t_2)/2}\right]_q \end{pmatrix} \\ & - \frac{1}{2\hbar}\int dt_1 dt_2 \left(\left[e^{i\phi(t_1)/2}\right]_c \; \left[e^{i\phi(t_1)/2}\right]_q \right) \begin{pmatrix} 0 & \beta^A \\ \beta^R & \beta^K \end{pmatrix}_{(t_1-t_2)} \begin{pmatrix} \left[e^{i\phi(t_2)/2}\right]_c \\ \left[e^{i\phi(t_2)/2}\right]_q \end{pmatrix} \\ & - \frac{1}{2\hbar}\int dt_1 dt_2 \left(\left[e^{-i\phi(t_1)/2}\right]_c \; \left[e^{-i\phi(t_1)/2}\right]_q \right) \begin{pmatrix} 0 & \beta^A \\ \beta^R & \beta^K \end{pmatrix}_{(t_1-t_2)} \begin{pmatrix} \left[e^{-i\phi(t_2)/2}\right]_c \\ \left[e^{-i\phi(t_2)/2}\right]_q \end{pmatrix} . \end{aligned} \tag{28}$$

We split

$$\mathcal{S}_{AES} = \mathcal{S}_\alpha^R + \mathcal{S}_\beta^R + \mathcal{S}_\alpha^K + \mathcal{S}_\beta^K \; . \tag{29}$$

The components are given by

$$i\mathcal{S}_\alpha^R = \frac{4}{\hbar}\int dt_1 dt_2 \, \alpha^R(t_1,t_2) \, \sin\left[\frac{\phi_c(t_1)-\phi_c(t_2)}{2\sqrt{2}}\right] \sin\frac{\phi_q(t_1)}{2\sqrt{2}} \cos\frac{\phi_q(t_2)}{2\sqrt{2}} \; . \tag{30}$$

$$i\mathcal{S}_\alpha^K = -\frac{2}{\hbar}\int dt_1 dt_2 \, \alpha^K(t_1,t_2) \, \cos\left[\frac{\phi_c(t_1)-\phi_c(t_2)}{2\sqrt{2}}\right] \sin\frac{\phi_q(t_1)}{2\sqrt{2}} \sin\frac{\phi_q(t_2)}{2\sqrt{2}} \; . \tag{31}$$

Analogously

$$i\mathcal{S}_\beta^R = \frac{4}{\hbar}\int dt_1 dt_2 \, \beta^R(t_1,t_2) \, \sin\left[\frac{\phi_c(t_1)+\phi_c(t_2)}{2\sqrt{2}}\right] \sin\frac{\phi_q(t_1)}{2\sqrt{2}} \cos\frac{\phi_q(t_2)}{2\sqrt{2}} \; . \tag{32}$$

$$i\mathcal{S}_\beta^K = \frac{2}{\hbar}\int dt_1 dt_2 \, \beta^K(t_1,t_2) \, \cos\left[\frac{\phi_c(t_1)+\phi_c(t_2)}{2\sqrt{2}}\right] \sin\frac{\phi_q(t_1)}{2\sqrt{2}} \sin\frac{\phi_q(t_2)}{2\sqrt{2}} \; . \tag{33}$$

(note the sign change in $\mathcal{S}_\beta^K$). Here

$$\begin{aligned} \alpha^R(t_1-t_2) \; = & \; \mathrm{Tr}\left[g_L^R(t_1-t_2)\,\hat{t}^\dagger\, g_R^K(t_2-t_1)\,\hat{t}\right] \\ & + \; \mathrm{Tr}\left[g_L^K(t_1-t_2)\,\hat{t}^\dagger\, g_R^A(t_2-t_1)\,\hat{t}\right] \; , \end{aligned} \tag{34}$$

and

$$\begin{aligned} \beta^R(t_1-t_2) \; = & \; -\mathrm{Tr}\left[f_L^R(t_1-t_2)\,\hat{t}^\dagger\, \tilde{f}_R^K(t_2-t_1)\,\hat{t}\right] \\ & - \; \mathrm{Tr}\left[f_L^K(t_1-t_2)\,\hat{t}^\dagger\, \tilde{f}_R^A(t_2-t_1)\,\hat{t}\right] \; . \end{aligned} \tag{35}$$

Note, that there is no trace over spin indexes here.