Selected topics in solid state physics 2

Part 7

Alexander Shnirman

on sabbatical leave from Karlsruhe Institute of Technology, Karlsruhe, Germany

(Dated: January 12, 2022)

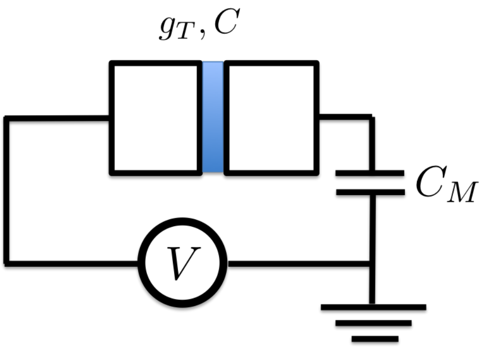

FIG. 1: FCS.

I. FULL COUNTING STATISTICS

There is a (poor man's) version of full counting statistics. Consider the scheme presented in Fig. 1. In addition to the tunnel junction we have a measurement device, which is a capacitor with capacitance C_M. The phase drop across the capacitance is denoted Φ. The Kirchhoff rule reads $\Phi + Vt - \Phi_J = 0$, where Φ_J is the phase drop across the tunnel junction (we choose Φ_J with the opposite orientation for convenience). This means that Φ is the only independent degree of freedom in the system, i.e., $\Phi_J = \Phi + Vt$. Upon integration out of the electrons in the leads the effective action reads

$$i\mathcal{S} = i\mathcal{S}_{AES}[\Phi(t) + Vt] + i\int_K dt \frac{C(\dot{\Phi} + V)^2}{2} + i\int_K dt \frac{C_M \dot{\Phi}^2}{2} \ . \tag{1}$$

This action allows to calculate the time evolution of the density matrix of the measurement capacitor C_M. Such a density matrix can be written in the phase representation $\rho_M(\Phi^+, \Phi^-)$ or in the charge representation $\rho_M(Q^+, Q^-)$. The operator of charge on C_M (more precisely on the island formed by the right lead and the capacitor C_M) reads $Q = -i\hbar\partial/\partial\Phi$. As usual

$$\rho_M(Q^+, Q^-) = \int d\Phi^+ d\Phi^- \, e^{-iQ^+\Phi^+/\hbar} \rho_M(\Phi^+, \Phi^-) e^{iQ^-\Phi^-/\hbar} \ . \tag{2}$$

With the help of the effective action we can propagate ρ_M in time:

$$\rho_M(\Phi_f^+, \Phi_f^-, t) = \Pi(t, \Phi_f^+, \Phi_f^-, \Phi_i^+, \Phi_i^-) \rho_M(\Phi_i^+, \Phi_i^-, 0) \ . \tag{3}$$

Here $\Pi(t) = \int D\Phi \exp[i\mathcal{S}]$ with the above specified boundary conditions.

Consider now the limit $C_M \to \infty$. In this regime of infinite mass the phase Φ cannot move, i.e., $\dot{\Phi} = 0$. Thus, the only path possible is $\Phi^+ = const.$ and $\Phi^- = const.$. The time evolution now reduces to

$$\rho_M(\Phi^+, \Phi^-, t) = \Pi(t, \Phi^+, \Phi^-)\rho_M(\Phi^+, \Phi^-, 0) \ . \tag{4}$$

It is still nontrivial. Imagine, the initial density matrix ($t = 0$) corresponded to diagonal density matrix in charge representation:

$$\begin{aligned}
\rho_i(\Phi^+, \Phi^-, 0) &= \int \frac{dQ^+dQ^-}{(2\pi\hbar)^2}\, e^{iQ^+\Phi^+/\hbar}\rho_i(Q^+, Q^-)e^{-iQ^-\Phi^-/\hbar} \\
&= \int \frac{dQ^+dQ^-}{(2\pi\hbar)^2}\, e^{iQ^+\Phi^+/\hbar}2\pi\hbar\delta(Q^+ - Q^-)\rho_i(Q^+)e^{-iQ^-\Phi^-/\hbar} \\
&= \int \frac{dQ}{2\pi\hbar}\, e^{iQ(\Phi^+-\Phi^-)/\hbar}\rho_i(Q) \equiv \rho_i(\Lambda) \ .
\end{aligned} \tag{5}$$

Here $\Lambda = \Phi^+ - \Phi^- = \sqrt{2}\Phi^q$ and Φ^q is the quantum component of Φ. Imagine, it so happens that the propagator Π is also a function only of $\Phi^+ - \Phi^-$. Then the final density matrix has the same property and we obtain

$$\rho_f(\Lambda) = \Pi(t, \Lambda)\rho_i(\Lambda) \ . \tag{6}$$

Performing the Fourier transform we obtain the convolution

$$\rho_f(Q) = \int \frac{dQ_1}{2\pi\hbar}\Pi(t, Q - Q_1)\rho_i(Q_1) \ . \tag{7}$$

Thus, the function $\Pi(t, Q)$ has the meaning of probability density that charge Q pass through the system (charge the capacitor C_M). The first moment is given by

$$\langle Q\rangle = \int \frac{dQ}{2\pi\hbar}\, Q\,\Pi(t, Q) = \int \frac{dQ}{2\pi\hbar}\, Q \int d\Lambda e^{-i\Lambda Q/\hbar}\Pi(\Lambda) = i\hbar\frac{\partial}{\partial\Lambda}\,\Pi(\Lambda)\Big|_{\Lambda\to 0} \ . \tag{8}$$

Analogously other moments and cumulants.

The generating function is then $\Pi(t, \Lambda)$. We obtain

$$\ln \Pi(t, \Lambda) = (i/\hbar)\mathcal{S}_{AES}[\Lambda] \ , \tag{9}$$

where for the normal tunnel junction

$$i\mathcal{S}_{AES} = -\frac{1}{\hbar}\int_K dt_1 dt_2\, \alpha(t_1, t_2)\, \cos\left[\frac{e(\Phi_J(t_1) - \Phi_J(t_2))}{\hbar}\right] \ . \tag{10}$$

Using the results obtained above we get $i\mathcal{S}_{AES} = i\mathcal{S}^R_{AES} + i\mathcal{S}^K_{AES}$ and

$$i\mathcal{S}^R_{AES} = \frac{2}{\hbar}\int dt_1 dt_2\, \alpha^R(t_1, t_2)\, \sin\left[\frac{\phi_{J,c}(t_1) - \phi_{J,c}(t_2)}{\sqrt{2}}\right] \sin\frac{\phi_{J,q}(t_1)}{\sqrt{2}} \cos\frac{\phi_{J,q}(t_2)}{\sqrt{2}} \ . \tag{11}$$

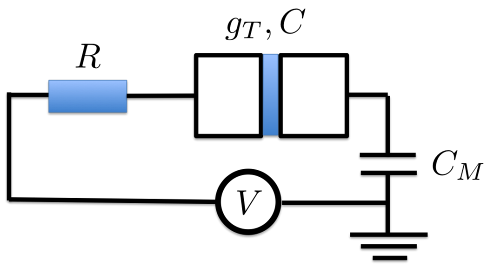

FIG. 2: Tunnel junction in series with a resistor.

$$i\mathcal{S}_{AES}^{K} = -\frac{1}{\hbar}\int dt_1 dt_2\, \alpha^K(t_1,t_2)\, \cos\left[\frac{\phi_{J,c}(t_1)-\phi_{J,c}(t_2)}{\sqrt{2}}\right] \sin\frac{\phi_{J,q}(t_1)}{\sqrt{2}} \sin\frac{\phi_{J,q}(t_2)}{\sqrt{2}}\ . \tag{12}$$

In our case, since $\Phi_J = \Phi + Vt$ and $\dot{\Phi}^+ = \dot{\Phi}^- = 0$ we obtain

$$i\mathcal{S}_{AES}^{R} = \frac{2}{\hbar}\int_0^t dt_1 \int_0^t dt_2\, \alpha^R(t_1,t_2)\, \sin\left[\frac{eV(t_1-t_2)}{\hbar}\right] \sin\frac{e\Lambda}{2\hbar}\cos\frac{e\Lambda}{2\hbar}\ . \tag{13}$$

$$i\mathcal{S}_{AES}^{K} = -\frac{1}{\hbar}\int dt_1 dt_2\, \alpha^K(t_1,t_2)\, \cos\left[\frac{eV(t_1-t_2)}{\hbar}\right] \sin\frac{e\Lambda}{2\hbar}\, \sin\frac{e\Lambda}{2\hbar}\ . \tag{14}$$

We observe that, indeed, the action depends only on Λ. This is not so in case of superconducting leads (Josephson junction).

We calculate the first moment, i.e., the charge that has passed. It is given by

$$\langle Q\rangle = It = i\hbar\frac{\partial}{\partial\Lambda}\,\Pi^R(\Lambda)\Big|_{\Lambda\to 0} = et\frac{1}{2\hbar^2}\left(\alpha^R(eV/\hbar) - \alpha^R(-eV/\hbar)\right)\ . \tag{15}$$

Using the properties of the function α we obtain

$$I = \frac{e}{2\hbar^2}\left[\alpha^>(eV/\hbar) - \alpha^<(eV/\hbar)\right]\ . \tag{16}$$

These are clearly currents flowing to the left and to the right. Using

$$\alpha^>(\omega) = \frac{1}{2}(\alpha^K + \alpha^R - \alpha^A) = \frac{\hbar^2}{e^2}\frac{\hbar\omega}{R_T}\left(\coth\frac{\hbar\omega}{2k_BT} + 1\right)\ , \tag{17}$$

$$\alpha^<(\omega) = \frac{1}{2}(\alpha^K - \alpha^R + \alpha^A) = \frac{\hbar^2}{e^2}\frac{\hbar\omega}{R_T}\left(\coth\frac{\hbar\omega}{2k_BT} - 1\right)\ , \tag{18}$$

where $R_T = 2\pi\hbar/(e^2 g_T)$, we obtain $I = V/R_T$.

II. $P(E)$-THEORY

Consider now the setup shown in Fig. 2. The phases on the impedance Φ_z, on the junction Φ_J, and on the measurement capacitance Φ are related as $\Phi_J = \Phi_z + \Phi + Vt$. As above, the phase Φ is not really dynamical, since $C_M \to \infty$. However, now, one of the phases is still dynamical and can fluctuate. This is Φ_z. Thus we obtain

$$\Pi(t,\Phi^+,\Phi^-) = \int D\Phi_z \exp\left[(i/\hbar)\mathcal{S}\right] \ , \tag{19}$$

where

$$i\mathcal{S} = i\mathcal{S}_Z[\Phi_z] + i\mathcal{S}_{AES}[\Phi+\Phi_z+Vt] + \oint_K dt \frac{C(\dot{\Phi}_z+V)^2}{2} \ . \tag{20}$$

The first part of this action is the action of a linear resistor (impedance), which reads

$$\begin{aligned} \mathcal{S}_Z &= -\frac{1}{2\hbar}\oint_K dt_1 \oint_K dt_2\ \Phi_z(t_1)\alpha_R(t_1,t_2)\,\Phi_z(t_2) \\ &= -\frac{1}{2\hbar}\int dt_1 \int dt_2 \ \begin{pmatrix} \Phi_{z,c}(t_1) & \Phi_{z,q}(t_1) \end{pmatrix} \begin{pmatrix} 0 & \alpha_R^A \\ \alpha_R^R & \alpha_R^K \end{pmatrix}_{(t_1-t_2)} \begin{pmatrix} \Phi_{z,c}(t_2) \\ \Phi_{z,q}(t_2) \end{pmatrix} \ , \end{aligned} \tag{21}$$

where

$$\mathrm{Re}\,\alpha_R^R(\omega) = \frac{\hbar\omega}{R} \ , \tag{22}$$

Note that $\mathrm{Im}\,\alpha_R^R(\omega)$ has to be dropped, as it represents an infinite renormalisation of the potential.

Assume now that the tunnelling conductance of the junctions is much smaller than the conductance of the linear resistance, $R_T \gg R$. Then we can expand:

$$\Pi(t,\Phi^+,\Phi^-) \approx \int D\Phi_z \left[1 + (i/\hbar)\mathcal{S}_{AES}\right] \exp\left[(i/\hbar)\mathcal{S}_0\right] \ , \tag{23}$$

where $\mathcal{S}_0 = \mathcal{S} - \mathcal{S}_{AES}$. The first leading term does not contain the counting fields and, thus, can be disregarded.

We use

$$i\mathcal{S}_{AES}[\Phi] = -\frac{1}{\hbar}\int_K dt_1 dt_2\, e^{-i(e/\hbar)\Phi(t_1)}\,\alpha(t_1,t_2)\,e^{i(e/\hbar)\Phi(t_2)} \ . \tag{24}$$

Thus we obtian

$$\begin{aligned} &\Pi(t,\Phi^+,\Phi^-) \approx 1 - \frac{1}{\hbar^2}\int D\Phi_z \ \exp\left[(i/\hbar)\mathcal{S}_0\right] \\ &\times \int_K dt_1 dt_2\, e^{-i(e/\hbar)[\Phi(t_1)+\Phi_z(t_1)+Vt_1]}\,\alpha(t_1,t_2)\,e^{i(e/\hbar)[\Phi(t_2)+\Phi_z(t_2)+Vt_2]} \ . \end{aligned} \tag{25}$$

The path integral over the phase of the impedance Φ_z reads

$$P(t_1,t_2) = \int D\Phi_z \ \exp\left[(i/\hbar)\mathcal{S}_0\right] e^{-i(e/\hbar)[\Phi_z(t_1)-\Phi_z(t_2)]} \ . \tag{26}$$

This is a Gaussian integral, thus

$$P(t_1,t_2) = \exp\left[-\frac{1}{2}(e/\hbar)^2 \langle T_K \left[\Phi_z(t_1) - \Phi_z(t_2)\right]^2 \rangle_0\right] \ , \tag{27}$$

where the averaging is performed with the action $\mathcal{S}_0$. We obtain

$$\Pi(t,\Phi^+,\Phi^-) \approx 1 - \frac{1}{\hbar^2}\int_K dt_1 dt_2\, e^{-i(e/\hbar)[\Phi(t_1)+Vt_1]}\,\tilde{\alpha}(t_1,t_2)\, e^{i(e/\hbar)[\Phi(t_2)+Vt_2]} \ , \tag{28}$$

where

$$\tilde{\alpha}(t_1,t_2) = \alpha(t_1,t_2)P(t_1,t_2) \ . \tag{29}$$

In full analogy with (13) and (14) we get $\Pi(t,\Lambda) = 1 + \Pi^R(t,\Lambda) + \Pi^K(t,\Lambda)$, where

$$\Pi^R(t,\Lambda) \ = \ \frac{2}{\hbar^2}\int_0^t dt_1 \int_0^t dt_2\, \tilde{\alpha}^R(t_1,t_2)\, \sin\left[\frac{eV(t_1-t_2)}{\hbar}\right] \sin\frac{e\Lambda}{2\hbar}\cos\frac{e\Lambda}{2\hbar} \ . \tag{30}$$

$$\Pi^K(t,\Lambda) = -\frac{1}{\hbar^2}\int dt_1 dt_2\, \tilde{\alpha}^K(t_1,t_2)\, \cos\left[\frac{eV(t_1-t_2)}{\hbar}\right] \sin\frac{e\Lambda}{2\hbar}\sin\frac{e\Lambda}{2\hbar} \ . \tag{31}$$

The first moment is given by

$$\langle Q\rangle = It = i\hbar\frac{\partial}{\partial\Lambda}\,\Pi^R(\Lambda)\Big|_{\Lambda\to 0} = et\frac{1}{2\hbar^2}\left(\tilde{\alpha}^R(eV/\hbar) - \tilde{\alpha}^R(-eV/\hbar)\right) \ . \tag{32}$$

Using the properties of the function $\tilde{\alpha}$ (the same as those of α) we obtain

$$I = \frac{e}{2\hbar^2}\left[\tilde{\alpha}^>(eV/\hbar) - \tilde{\alpha}^<(eV/\hbar)\right] \ . \tag{33}$$

Further we observe

$$\tilde{\alpha}^>(t) = \alpha^>(t)P^>(t) \quad , \quad \tilde{\alpha}^<(t) = \alpha^<(t)P^<(t) \ . \tag{34}$$

Further

$$P^>(t) = \exp\left[(e/\hbar)^2\,\langle\Phi_z(t)(\Phi_z(0)-\Phi_z(t))\rangle_0\right] \ , \tag{35}$$

$$P^<(t) = \exp\left[(e/\hbar)^2\,\langle(\Phi_z(0)-\Phi_z(t))\Phi_z(t)\rangle_0\right] \ . \tag{36}$$

What is left is to calculate the correlation functions $J^>(t) = \langle\Phi_z(t)\Phi_z(0)\rangle_0$ and $J^<(t) = \langle\Phi_z(0)\Phi_z(t)\rangle_0$. For the current we thus obtain

$$I = \frac{e}{2\hbar^2}\int\frac{d\omega}{2\pi}\left[\alpha^>(eV/\hbar-\omega)P^>(\omega) - \alpha^<(eV/\hbar-\omega)P^<(\omega)\right] \ . \tag{37}$$

Looking at this expression and comparing to (16) one understands the physical meaning of the correlation functions $P^>$ and $P^<$. They give the probability density for energy $\hbar\omega$ to be absorbed by the environment (linear resistor in our case).

A. Calculating functions $J^>$ and $J^<$

The action governing fluctuations of Φ_z reads

$$i\mathcal{S}_0 = i\mathcal{S}_Z[\Phi_z] + \oint_K dt \frac{C\dot{\Phi}_z^2}{2} \ , \tag{38}$$

where we have dropped the full derivative from the action. This can be rewritten as

$$\begin{aligned} \mathcal{S}_0 &= -\frac{1}{2\hbar}\oint_K dt_1 \oint_K dt_2 \ \Phi_z(t_1)\alpha_{RC}(t_1,t_2)\,\Phi_z(t_2) \\ &= -\frac{1}{2\hbar}\int dt_1 \int dt_2 \ \begin{pmatrix} \Phi_{z,c}(t_1) & \Phi_{z,q}(t_1) \end{pmatrix} \begin{pmatrix} 0 & \alpha_{RC}^A \\ \alpha_{RC}^R & \alpha_{RC}^K \end{pmatrix}_{(t_1-t_2)} \begin{pmatrix} \Phi_{z,c}(t_2) \\ \Phi_{z,q}(t_2) \end{pmatrix} \ , \end{aligned} \tag{39}$$

where

$$\alpha_{RC}^R(\omega) = \mathrm{Re}\alpha_R^R(\omega) - i\hbar C\omega^2 = \frac{\hbar\omega}{R} - i\hbar C\omega^2 = \hbar\omega\left(\frac{1}{R} - iC\omega\right) \ . \tag{40}$$

$$\alpha_{RC}^K(\omega) = \left(\alpha_{RC}^R(\omega) - \alpha_{RC}^A(\omega)\right)\coth\frac{\hbar\omega}{2k_BT} = \frac{2\hbar\omega}{R}\ \coth\frac{\hbar\omega}{2k_BT} \ . \tag{41}$$

Inverting the matrix we obtain

$$\begin{pmatrix} J^K & J^R \\ J^A & 0 \end{pmatrix}_\omega = \hbar^2 \begin{pmatrix} 0 & \alpha_{RC}^A \\ \alpha_{RC}^R & \alpha_{RC}^K \end{pmatrix}_\omega^{-1} = \hbar^2 \begin{pmatrix} -[\alpha_{RC}^R]^{-1}\alpha_{RC}^K[\alpha_{RC}^A]^{-1} & [\alpha_{RC}^R]^{-1} \\ [\alpha_{RC}^A]^{-1} & 0 \end{pmatrix}_\omega \tag{42}$$

We obtain

$$J^>(\omega) = \frac{1}{2}(J^K + J^R - J^A) = \frac{\hbar\mathrm{Re}Z_t(\omega)}{\omega}\left(\coth\frac{\hbar\omega}{2k_BT} + 1\right) \ , \tag{43}$$

$$J^<(\omega) = \frac{1}{2}(J^K - J^R + J^A) = \frac{\hbar\mathrm{Re}Z_t(\omega)}{\omega}\left(\coth\frac{\hbar\omega}{2k_BT} - 1\right) \ , \tag{44}$$

where

$$\mathrm{Re}Z_t(\omega) = \frac{R}{(1 + C^2R^2\omega^2)} \ . \tag{45}$$

The impedance

$$Z_t^{-1} = \frac{1}{R} - iC\omega \tag{46}$$

is the total impedance as seen from the junction. Further

$$\begin{aligned} \ln P^>(t) &= (e/\hbar)^2(J^>(t) - J^>(0)) = (e/\hbar)^2\int\frac{d\omega}{2\pi}\left[e^{-i\omega t} - 1\right] J^>(\omega) \\ &= (e/\hbar)^2\int\frac{d\omega}{2\pi}\frac{\hbar\mathrm{Re}Z_t(\omega)}{\omega}\left(\coth\frac{\hbar\omega}{2k_BT}[\cos(\omega t) - 1] - i\sin(\omega t)\right) \\ &= \int\frac{d\omega}{\omega}\frac{\mathrm{Re}Z_t(\omega)}{R_K}\left(\coth\frac{\hbar\omega}{2k_BT}[\cos(\omega t) - 1] - i\sin(\omega t)\right) \ . \end{aligned} \tag{47}$$

$$
\begin{aligned}
\ln P^{<}(t) &= (e/\hbar)^2 (J^{<}(t) - J^{<}(0)) = (e/\hbar)^2 \int \frac{d\omega}{2\pi} \left[e^{-i\omega t} - 1 \right] J^{<}(\omega) \\
&= (e/\hbar)^2 \int \frac{d\omega}{2\pi} \frac{\hbar \mathrm{Re} Z_t(\omega)}{\omega} \left(\coth \frac{\hbar\omega}{2k_B T} \left[\cos(\omega t) - 1\right] + i \sin(\omega t) \right) \\
&= \int \frac{d\omega}{\omega} \frac{\mathrm{Re} Z_t(\omega)}{R_K} \left(\coth \frac{\hbar\omega}{2k_B T} \left[\cos(\omega t) - 1\right] + i \sin(\omega t) \right) \quad . \qquad (48)
\end{aligned}
$$

Further information on $P(E)$ theory in Ref. [1].

[1] G.-L. Ingold and Y. V. Nazarov, in *Single Charge Tunneling, NATO ASI Series B, Vol. 294, arXiv:cond-mat/0508728*, edited by H. Grabert and M. H. Devoret (Plenum Press, New York, 1992), chap. 2, pp. 21–107.