Gravity 1 - Recitation 1

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1 Minkowski Spacetime

In Albert Einstein's original treatment (1905), *the special theory of relativity* is based on two postulates:

1. The laws of physics are invariant (that is, identical) in all inertial frames of reference (that is, frames of reference with no acceleration).

2. The speed of light in vacuum is the same for all observers in different inertial frames.

We will start from a "modern" point of view, asserting *Minkowski spacetime.*

1.1 The line element

Spacetime can be assigned with four coordinates $x^\mu = \left(x^0, x^1, x^2, x^3\right)$. Inertial Cartesian coordinates are $\left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z)$, where c is the speed of light. These spacetime coordinates define an inertial frame. The *line element* ds^2 of Minkowski spacetime in inertial Cartesian coordinates takes the form

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

Line element of Minkowski spacetime in Cartesian coordinates

The **value** of a spacetime distance (squared) $(\Delta s)^2$ between two points in spacetime is independent of the choice of coordinates. However, the **form** of the line element, i.e., the formula how to compute it, does depend on the choice of coordinates. For example, in the Euclidean plane $\mathbb{R}^2$, the line element in Cartesian coordinates (x, y) has the form $ds^2_{Euclidean} = dx^2 + dy^2$, while in polar coordinates (r, θ) it has the form $ds^2_{polar} = dr^2 + r^2 d\theta^2$. Nevertheless, $ds^2_{Euclidean} = ds^2_{polar}$.

An inertial frame is a choice of coordinates for spacetime such that ds^2 has the form (1), and therefore another reference frame specified by coordinates $x^{\mu'} = (t', x', y', z')$ is inertial if also $ds^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$.

1.2 The invariance of the speed of light

Let us check that this approach respect the second postulate. We start in an inertial frame x^μ. A light moving along the positive x direction has velocity $\frac{dx}{dt} = c$, and $\frac{dy}{dt} = \frac{dz}{dt} = 0$. Therefore

$$dx = cdt \tag{2}$$

and of course $dy = dz = 0$. What is the infinitesimal spacetime distance of the light ray? Plug (2) into (1) we find

$$ds^2 = -c^2 dt^2 + dx^2 = -c^2 dt^2 + (cdt)^2 = 0 \tag{3}$$

Light moves in spacetime along trajectories with zero spacetime length. We say that light has *null worldline*. In another inertial frame $x^{\mu'}$, for simplicity with $y' = y$, $z' = z$, ds^2 has the same value (zero) and the same form

$$ds^2 = -c^2 dt'^2 + dx'^2 = 0 \tag{4}$$

Therefore

$$\frac{dx'}{dt'} = \pm c \tag{5}$$

We found that indeed light has the same speed in both frames. The minus sign here may appear due to inversion of the t or the x coordinates. We will exclude these kind of discrete transformations, see section 2.5.

2 Lorentz Transformations

We **define** *Lorentz transformations* as the linear coordinate transformations that preserve the infinitesimal length form (1). These include both rotations of space and boosts. The non-linear coordinate transformations which preserve the form of (1) are translations: $x^\mu \to x^\mu + a^\mu$, where a^μ is constant. Since $dx^{\mu'} = dx^\mu$ the form of (1) is conserved.

2.1 The Minkowski metric tensor

Let us write (1) in a shorthanded way as (a quadratic form)

$$ds^2 = \begin{pmatrix} dx^0 & dx^1 & dx^2 & dx^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} = (dx)^T \eta \, (dx) \tag{6}$$

where

$$dx^\mu = \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \tag{7}$$

and

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

is the *Minkowski metric tensor* in Cartesian coordinates. The one minus sign in the metric is a *Lorentzian signature.*

In index notation, using the *Einstein summation convention* (6) reads

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{9}$$

Line element - Minkowski metric

So, we are looking for linear transformations that leave the form of (6) invariant. This is identical to require that the transformation leaves η invariant (have the same components in the new coordinates).

2.2 Finding Lorentz

We denote the coordinate transformation in matrix form as

$$x \to x' = \Lambda x \tag{10}$$

where Λ is the Lorentz transformation matrix. We transform (6) and require $\eta' = \eta$

$$\begin{aligned} ds^2 &= (dx)^T \eta \, (dx) = (dx')^T \eta' \, (dx') \\ &\overset{!}{=} (dx')^T \eta \, (dx') = (dx)^T \Lambda^T \eta \Lambda \, (dx) \end{aligned} \tag{11}$$

Therefore

$$\Lambda^T \eta \Lambda = \eta \tag{12}$$

The Minkowski metric tensor is invariant under Lorentz transformations

In index notation, (10) and (12) read

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \tag{13}$$

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho} \Lambda^{\nu'}{}_{\sigma} \eta_{\mu'\nu'} \tag{14}$$

2.2.1 Rotations

Pure spatial transformations that leave lengths invariant are spatial rotations[1]. Such matrices R are called orthogonal matrices $R \in O(3)$ and they obey

$$R^T I R = I \tag{15}$$

where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Orthogonality (15) implies

$$|detR| = 1 \tag{16}$$

. (15) is just the spatial part of (12). So pure spatial Lorentz transformations are just spatial rotations. For example, a rotation of angle θ in the xy plane is

$$(\Lambda_{xy})^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

2.2.2 Discussion

Before we move on to find Lorentz transformations involving the time axis, let us have a closer look at the defining equation they need to obey (12). We take

[1]These include both proper and improper rotations.

the determinant of equation (12)

$$\begin{aligned} det\left(\eta\right) = det\left(\Lambda^{T}\eta\Lambda\right) \\ =&det\left(\Lambda^{T}\right)det\left(\eta\right)det\left(\Lambda\right) \\ =&det\left(\Lambda\right)det\left(\eta\right)det\left(\Lambda\right) \\ &\Downarrow \\ 1 = \left(det\Lambda\right)^{2} \end{aligned} \tag{18}$$

We find, as expected,

$$|det\left(\Lambda\right)| = 1 \tag{19}$$

Notice that in the derivation (18) the determinant of the metric η was canceled on both sides, so (19) does not take into account the signature of the metric. Therefore condition (19) by itself does not exclude a simple rotation in the tx plane also. There is actually a second general condition we can conclude, which does address the signature. For this we look only on the time-time component of (12). We do this by setting $(\rho,\sigma) = (0,0)$ in the index notation version (14)

$$\begin{aligned} -1 = \eta_{00} = &\Lambda^{\mu'}{}_{0}\Lambda^{\nu'}{}_{0}\eta_{\mu'\nu'} \\ =&\Lambda^{0'}{}_{0}\Lambda^{0'}{}_{0}\eta_{0'0'} + \Lambda^{1'}{}_{0}\Lambda^{1'}{}_{0}\eta_{1'1'} + \Lambda^{2'}{}_{0}\Lambda^{2'}{}_{0}\eta_{2'2'} + \Lambda^{3'}{}_{0}\Lambda^{3'}{}_{0}\eta_{3'3'} \\ =&-\left(\Lambda^{0'}{}_{0}\right)^{2} + \left(\Lambda^{1'}{}_{0}\right)^{2} + \left(\Lambda^{2'}{}_{0}\right)^{2} + \left(\Lambda^{3'}{}_{0}\right)^{2} \end{aligned} \tag{20}$$

$$\Downarrow$$

$$\left(\Lambda^{0'}{}_{0}\right)^{2} = 1 + \left(\Lambda^{1'}{}_{0}\right)^{2} + \left(\Lambda^{2'}{}_{0}\right)^{2} + \left(\Lambda^{3'}{}_{0}\right)^{2} \geq 1 \tag{21}$$

$$\Downarrow$$

$$|\Lambda^{0'}{}_{0}| \geq 1 \tag{22}$$

A rotation in the tx plane would have $|\Lambda^{0'}{}_{0}| = |cos\theta| \leq 1$, just the opposite of condition (22). An Euclidean signature of the metric indeed would have yield $|\Lambda^{0'}{}_{0}| \leq 1$. We want some kind of "rotation" to fulfill condition (19) but also condition (22). Hyperbolic cosine has the property that $cosh\phi \geq 1$, and we can cook up a determinant of the form $cosh^2\phi - sinh^2\phi = 1$!

2.2.3 Boosts

A "rotation" in the tx plane would need to take into account the minus sign in the time component of η. Thus a "rotation" in the tx plane is a "hyperbolic rotation", of hyperbolic angle ϕ, also called *rapidity*. This transformation is the so-called boost in the x axis. It reads

$$(\Lambda_{tx})^{\mu'}_{\ \nu} = \begin{pmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

Boost is hyperbolic rotation in tx plane

Notice that the hyperbolic functions are not periodic, nor bound, and the hyperbolic angle range is $-\infty < \phi < \infty$. Λ_{ty} and Λ_{tz} can be written in the same fashion.

First, we check condition (19)

$$det\Lambda_{tx} = \begin{vmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} cosh\phi & -sinh\phi \\ -sinh\phi & cosh\phi \end{vmatrix} = cosh^2\phi - sinh^2\phi = 1 \tag{24}$$

and condition (22)

$$(\Lambda_{tx})^{0'}_{\ 0} = cosh\phi \geq 1 \tag{25}$$

Now, let us check that Λ_{tx} indeed leaves η invariant.

$$\begin{aligned}
\left(\Lambda_{tx}^T \eta \Lambda_{tx}\right)_{\mu\nu} &= \begin{pmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -cosh\phi & sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -cosh^2\phi + sinh^2\phi & 0 & 0 & 0 \\ 0 & -sinh^2\phi + cosh^2\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\mu\nu} \qquad (26)
\end{aligned}$$

where we used the hyperbolic identity

$$cosh^2\phi - sinh^2\phi = 1 \qquad (27)$$

Explicitly, the boost $x' = \Lambda_{tx} x$ is

$$\begin{aligned}
ct' &= (cosh\phi)\, ct - (sinh\phi)\, x \\
x' &= (cosh\phi)\, x - (sinh\phi)\, ct \\
y' &= y \\
z' &= z \qquad (28)
\end{aligned}$$

As an exercise, let us also show explicitly that $ds^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 = -cdt^2 + dx^2 + dy^2 + dz^2$. Since $dy' = dy$ and $dz' = dz$ we concentrate on the t, x coordinates.

$$
\begin{aligned}
-c^2dt'^2 + dx'^2 =& -\left((cosh\phi)\,cdt - (sinh\phi)\,dx\right)^2 + \left((cosh\phi)\,dx - (sinh\phi)\,cdt\right)^2 \\
=& \left(-cosh^2\phi + sinh^2\phi\right)(cdt)^2 + \left(-sinh^2\phi + cosh^2\phi\right)dx^2 \\
=& -c^2dt^2 + dx^2
\end{aligned} \tag{29}
$$

2.3 Relation to 3-velocity

From (28) we see that the point $x' = 0$ is moving with 3-velocity v relative to the unprimed reference frame

$$
\beta = \frac{v}{c} = \frac{x}{ct} = tanh\phi \tag{30}
$$

$$
-1 < \beta < 1
$$

Define γ as

$$
\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}} = \left(1 - tanh^2\phi\right)^{-\frac{1}{2}} = cosh\phi \tag{31}
$$

$$
1 \le \gamma < \infty \tag{32}
$$

Plug (30) and (31) into (28), we get the familiar boost transformation

$$
\begin{aligned}
ct' &= \gamma\,(ct - \beta x) \\
x' &= \gamma\,(x - \beta ct)
\end{aligned} \tag{33}
$$

and the Lorentz matrix Λ_{tx} is

$$
(\Lambda_{tx})^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

2.4 Boost in spacetime diagram

Under hyperbolic rotation, the new t' axis ($x' = 0$) is the line $ct = \frac{1}{tanh\phi}x$ and the new x' axis ($t' = 0$) is the line $ct = (tanh\phi)\,x$. We therefore see that the space and time axes are rotated into each other by angle ϕ, in contrast to a regular rotation. Nonetheless, they remain orthogonal in the spacetime sense, i.e., with respect to the Minkowski metric.

As a regular rotation moves a point along a circle, so a hyperbolic rotation moves a point along a hyperbola. Therefore boost moves a point on a hyperbola on spacetime diagram.

2.5 Restricting Lorentz

2.5.1 $O(3)$ group components

The orthogonal group $O(3)$ consists of two disconnected components: *proper rotations* with $det(R) = +1$, and *improper rotations* with $det(R) = -1$. The improper rotations not only rotate, but also make a *parity* transformation, flipping all the spatial axes ($x \to -x, y \to -y, z \to -z$). Parity matrix is $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Transformation that flips odd number of axis has determinant -1. It changes the *orientation* of space from right handed to left handed (and vise versa). Flipping two axes on the other hand is identical to a rotation of 180°.

The subgroup $SO(3)$ consists only the proper rotations, i.e., it is the component including the identity element and all transformations that can be reached continuously from it, by varying the three angle parameters $(\theta_1, \theta_2, \theta_3)$. It lacks parity transformation.

2.5.2 $O(3,1)$ group components

For the Lorentz group we found two binary distinctions between the Lorentz transformations: $|det(\Lambda)| = 1$ and $|\Lambda^{0'}_{\ 0}| \geq 1$. Since $\Lambda^{0'}_{\ 0} \geq 1$ or $\Lambda^{0'}_{\ 0} \leq -1$, elements with positive/negative $\Lambda^{0'}_{\ 0}$ cannot be continuously reached to each other by some parameter. [2] Negative values include *time reversal* ($t \to -t$). Time reversal matrix is $T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Therefore, the Lorentz group consists of four disconnected components, summarized with their names in the following table:

	$det\Lambda = +1$	$det\Lambda = -1$
$\Lambda^{0'}_{\ 0} \geq 1$	Proper Orthochronous	Improper Orthochronous
$\Lambda^{0'}_{\ 0} \leq -1$	Proper Antichronous	Improper Antichronous

[2]It does not happen in the Euclidean case, since there $-1 \leq \Lambda^{0'}_{\ 0} \leq 1$.

Subgroups:

• The *Lorentz Group* is the Pseudo-Orthogonal Group $O(3,1)$. It consists all linear transformations that leave the Minkowski metric invariant.

• The *Proper Lorentz Group* is the Special Pseudo-Orthogonal Group, denoted $SO(3,1)$. It consists only the Lorentz transformations with $det\Lambda = +1$.

• The *Restricted Lorentz Group* is the *Proper Orthochronous Lorentz Group*, denoted $SO^{+}(3,1)$. It consists only the Lorentz transformations with $det\Lambda = +1$ and $\Lambda^{0'}_{\ 0} \geq 1$. It includes the identity element, and all transformations that can be reached continuously from it, by varying the six angle parameters $(\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3)$. It lacks time reversal and parity transformations.

We work with the restricted Lorentz group.