Gravity 1 - Recitation 2

October 30, 2021

Contents

1 Phenomena

1.1 Time Dilation

An observer at rest is located at some fixed position x_0, therefore $\Delta x = 0$. He measures some time interval Δt. Another observer $x^{\mu'}$ is traveling at constant speed v along the x - direction. What time interval $\Delta t'$ would he measure?

Make a Lorentz transformation for the time component

$$c\Delta t' = \gamma \left(c\Delta t - \beta \Delta x \right) \tag{1}$$

Therefore

Time dilation

$$\Delta t' = \gamma \Delta t \geq \Delta t \tag{2}$$

The clock of the moving observer ticks at a slower rate. $\Lambda^{0'}_{\ 0} = cosh\phi = \gamma \geq 1$ is called the *Lorentz factor*.

1.2 Length Contraction

An inertial observer in frame x^{μ} measures a length of $\Delta x = L$. Another observer $x^{\mu'}$ is traveling at constant speed v along the x - direction. In his frame he measures simultaneously ($\Delta t' = 0$) some length $\Delta x'$. Find $\Delta x'$.

Make an inverse Lorentz transformation for the x component

$$\Delta x = \gamma \left(\Delta x' + \beta \Delta t' \right) \tag{3}$$

Therefore

Length contraction

$$\Delta x' = \frac{L}{\gamma} \leq L \tag{4}$$

1.3 Exercise

An inertial observer in frame x^{μ} measures simultaneously ($\Delta t = 0s$) a length of $\Delta x = 3m$. Another observer $x^{\mu'}$ is traveling at constant speed v along the positive x - direction. In his frame he measures between the same two events the time interval $\Delta t' = 10^{-8}s$. The speed of light is $c = 3 \cdot 10^8 \frac{m}{s}$.

Find $\Delta x'$ in two ways: 1. using Lorentz transformations. 2. Using the invariant line element.

1.3.1 Solving with Lorentz transformations

We solve in two steps. First step: find γ by transforming the t component. Second step: find $\Delta x'$ by transforming the x component.

Find γ

$$c\Delta t' = \gamma\left(c\Delta t - \beta\Delta x\right) \tag{5}$$

plug in the data

$$1 = -\gamma\beta \tag{6}$$

$$1 = \gamma^2\beta^2 = \frac{\beta^2}{1-\beta^2} \tag{7}$$

$$\beta^2 = \frac{1}{2} \tag{8}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{1}{2}}} = \sqrt{2} \tag{9}$$

Find $\Delta x'$

$$\Delta x' = \gamma\left(\Delta x - \beta\Delta t\right) \tag{10}$$

$$\Delta x' = \gamma\Delta x = 3\sqrt{2} = \sqrt{18}\,[m] \tag{11}$$

1.3.2 Solving with invariant line element

We solve in two steps. First step: Compute $(\Delta s)^2$. Second step: Find $\Delta x'$ by demanding $(\Delta s')^2 = (\Delta s)^2$.

$$(\Delta s)^2 = -\left(c\Delta t\right)^2 + (\Delta x)^2 = 9\left[m^2\right] \tag{12}$$

$$(\Delta s)^2 = (\Delta s')^2 = -\left(c\Delta t'\right)^2 + (\Delta x')^2 \tag{13}$$

$$(\Delta x')^2 = (\Delta s)^2 + \left(c\Delta t'\right)^2 = 9 + 3^2 = 18\left[m^2\right] \tag{14}$$

$$\Delta x' = \sqrt{18}\,[m] \tag{15}$$

2 Transformation of 3-velocity

2.1 Boost along the x - direction

In an inertial frame x^μ a particle is moving with a 3-velocity $u^i = \left(u^1, u^2, u^3\right)$ (Cartesian coordinates). Another inertial frame $x^{\mu'}$ is moving with relative constant velocity v along the positive x - direction. Let us find the 3-velocity $u^{i'}$ in the second inertial frame.

The 3-velocities in the x^μ frame are

$$u^i = \frac{dx^i}{dt} \tag{16}$$

and the 3-velocities in the $x^{\mu'}$ frame are

$$u^{i'} = \frac{dx^{i'}}{dt'} \tag{17}$$

Notice that the transformed velocity is influenced both by the space coordinate transformation and the time coordinate transformation.

The transformation of the velocity in the direction of the moving frame is

$$u^{1'} = \frac{dx'}{dt'} = \frac{\gamma\left(dx - \beta c dt\right)}{\gamma\left(dt - \frac{\beta}{c}dx\right)} = \frac{\frac{dx}{dt} - \beta c}{1 - \frac{\beta}{c}\frac{dx}{dt}} = \frac{u^1 - v}{1 - \frac{u^1 v}{c^2}} \tag{18}$$

The transformation of a velocity perpendicular to the moving frame is (for example $i = 2$. It is the same for $i = 3$)

$$u^{2'} = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{\beta}{c}dx\right)} = \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{u^2}{\gamma\left(1 - \frac{u^1 v}{c^2}\right)} \tag{19}$$

The formulae are

$$u^{1'} = \frac{u^1 - v}{1 - \frac{u^1 v}{c^2}} \tag{20}$$

$$u^{2'} = \frac{u^2}{\gamma\left(1 - \frac{u^1 v}{c^2}\right)} \tag{21}$$

$$u^{3'} = \frac{u^3}{\gamma\left(1 - \frac{u^1 v}{c^2}\right)} \tag{22}$$

3-velocity transformation for Boost along x-direction

Lets look at two special cases.

First case - a light moving in the x - direction, so $u^i = (c, 0, 0)$. Plug into (20)(21)(22) we find

$$u^{1'} = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c\,(c - v)}{c - v} = c \tag{23}$$

and $u^{2'} = u^{3'} = 0$. In the moving frame the light has the same speed and same direction.

Second case - a light moving in the y - direction, so $u^i = (0, c, 0)$. Plug into (20)(21)(22) we find

$$u^{1'} = -v \tag{24}$$

$$u^{2'} = \frac{c}{\gamma} \tag{25}$$

and $u^{3'} = 0$. (24) is like a Galilean transformation. The light has no velocity in the x direction in the original frame. In the frame moving with velocity v in the x direction it has velocity v in this opposite direction. A Galilean transformation would yield $u^{2'} = c$, yet (25) corrects this. The velocity in the y direction is reduced by the Lorentz factor, such that the **total speed** of light would not change.

$$\begin{aligned} |\boldsymbol{u'}|^2 = &\left(u^{1'}\right)^2 + \left(u^{2'}\right)^2 + \left(u^{3'}\right)^2 = v^2 + \frac{c^2}{\gamma^2} \\ =&c^2\left(\left(\frac{v}{c}\right)^2 + \gamma^{-2}\right) = c^2\left(\beta^2 + 1 - \beta^2\right) = c^2 \end{aligned} \tag{26}$$

2.2 Exercise

In a lab rest frame, a light ray is traveling in the xy plane in a direction of angle θ, counter clock-wise with respect to the x axis. An observer $x^{\mu'}$ is traveling at constant speed v along the positive x - direction.

1. Show that in the observer's frame the light ray has velocity c.
2. What is the angle θ' between the light ray and the x' axis?

2.2.1 Calculating the speed of light

Let us use units with $c = 1$.

The velocity in the lab frame is

$$u^i = (cos\theta, sin\theta, 0) \tag{27}$$

By (20)(21)(22), the velocity in the observer's frame is

$$u^{i'} = \left(\frac{cos\theta - v}{1 - vcos\theta}, \frac{sin\theta}{\gamma \left(1 - vcos\theta\right)}, 0 \right) \tag{28}$$

$$\begin{aligned}
|\boldsymbol{u'}|^2 = & \left(u^{1'}\right)^2 + \left(u^{2'}\right)^2 + \left(u^{3'}\right)^2 = \left(\frac{cos\theta - v}{1 - vcos\theta} \right)^2 + \left(\frac{sin\theta}{\gamma \left(1 - vcos\theta\right)} \right)^2 \\
= & \frac{1}{\left(1 - vcos\theta\right)^2} \left(cos^2\theta - 2vcos\theta + v^2 + \left(1 - v^2\right) sin^2\theta \right) \\
= & \frac{1}{\left(1 - vcos\theta\right)^2} \left(cos^2\theta + sin^2\theta - 2vcos\theta + v^2 \left(1 - sin^2\theta\right) \right) \\
= & \frac{1}{\left(1 - vcos\theta\right)^2} \left(1 - 2vcos\theta + v^2 cos^2\theta \right) = \frac{\left(1 - vcos\theta\right)^2}{\left(1 - vcos\theta\right)^2} = 1 = c^2
\end{aligned} \tag{29}$$

2.2.2 Calculating the angle

$$tan\theta' = \frac{u^{2'}}{u^{1'}} = \frac{sin\theta}{\gamma \left(cos\theta - v\right)} = \frac{\sqrt{1 - v^2} sin\theta}{cos\theta - v} \tag{30}$$

3 Lorentz Transformation of the Electromagnetic Field

The electromagnetic field tensor $F^{\mu\nu}$, also called the *field strength* tensor of the electromagnetic field, in an inertial frame is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{31}$$

It is an *antisymmetric* tensor

$$F^{\mu\nu} = -F^{\nu\mu} \tag{32}$$

What is the electric field $E^{i'}$ in another inertial frame, moving with relative velocity v along the x axis?

To find out, we transform the tensor $F^{\mu\nu}$ and look at the desired components.

A $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$-tensor transformation rule is

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}F^{\rho\sigma} \tag{33}$$

where

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

We work with units $c = 1$, so $\beta = v$. When making index calculation we notice which components vanish, so not to write the whole sum explicitly.

The component parallel to v transforms

$$\begin{aligned} E'_x &= F^{0'1'} = \Lambda^{0'}{}_{\rho}\Lambda^{1'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{0'}{}_{0}\Lambda^{1'}{}_{1}F^{01} + \Lambda^{0'}{}_{1}\Lambda^{1'}{}_{0}F^{10} \\ &= \gamma^2 E_x + \gamma^2 v^2 \left(-E_x\right) = E_x\gamma^2\left(1 - v^2\right) = E_x \end{aligned} \tag{35}$$

The y - component transforms

$$\begin{aligned} E'_y &= F^{0'2'} = \Lambda^{0'}{}_{\rho}\Lambda^{2'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{0'}{}_{0}\Lambda^{2'}{}_{2}F^{02} + \Lambda^{0'}{}_{1}\Lambda^{2'}{}_{2}F^{12} \\ &= \gamma E_y - \gamma v B_z = \gamma\left(E_y + v_z B_x - v_x B_z\right) \\ &= \gamma\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)_y \end{aligned} \tag{36}$$

The z - component transforms

$$\begin{aligned} E'_z &= F^{0'3'} = \Lambda^{0'}{}_{\rho}\Lambda^{3'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{0'}{}_{0}\Lambda^{3'}{}_{3}F^{03} + \Lambda^{0'}{}_{1}\Lambda^{3'}{}_{3}F^{13} \\ &= \gamma E_z - \gamma v\left(-B_y\right) = \gamma\left(E_z + v_x B_y - v_y B_x\right) \\ &= \gamma\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)_z \end{aligned} \tag{37}$$

where we used the fact that

$$v^i = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{38}$$

and

$$(\boldsymbol{v} \times \boldsymbol{B})^i = \begin{pmatrix} v_y B_z - v_z B_y \\ v_z B_x - v_x B_y \\ v_x B_y - v_y B_x \end{pmatrix} \tag{39}$$

We conclude, in general that

$$\boldsymbol{E}'_{\shortparallel} = \boldsymbol{E}_{\shortparallel} \tag{40}$$

$$\boldsymbol{E}'_{\perp} = \gamma \left(\boldsymbol{E}_{\perp} + \boldsymbol{v} \times \boldsymbol{B} \right) \tag{41}$$

Lorentz transformation of the electric field

where $\shortparallel$ and $\perp$ mean parallel and perpendicular to $\boldsymbol{v}$.