Gravity 1 - Recitation 3

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Contents

1 Four - Vectors

In order to study kinematics, we first review the concept of 4-vectors. Minkowski space is a four dimensional vector space equipped with a (pseudo) inner product. A vector in Minkowski space is called a *4-vector*.

1.1 The Metric Tensor Revisited (Math Supplement)

1.1.1 The metric tensor as an inner product

A *pseudo inner product* on a real vector space V is a map

$$(\ , \ ) : V \times V \to \mathbb{R} \tag{1}$$

that satisfies the following conditions:

1. Billinear

$$\begin{aligned} (av + bu, w) &= a\,(v, w) + b\,(u, w) \\ (w, av + bu) &= a\,(w, v) + b\,(w, u) \end{aligned} \tag{2}$$

2. Symmetric

$$(v, u) = (u, v) \tag{3}$$

3. Non - degenerate

$$if \ \ (v, u) = 0 \ \forall v \ \ \Rightarrow u = 0 \tag{4}$$

For a genuine real inner product (like the usual dot product), non-degeneracy is replaced by positive-definiteness, i.e., $(v, v) > 0$ for any non-zero vector and $(v, v) = 0$ only for the zero vector. A pseudo inner product is more general, non-zero vectors can have a zero norm, and also negative norm squared. The feature that still remains in (4) is that the zero vector is the only vector which is orthogonal to any other vector.

We can view the inner product in three ways: As a map "(,)" (as defined above); As a product "$\cdot$" ("scalar product"); Or as an object by itself (like we do with functions), name it "g". What kind of object would it be? Well, a billinear map that takes two vectors and returns a scalar is a $\binom{0}{2}$-tensor. Therefore, a

choice of a non-degenerate symmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$-tensor, called a *metric tensor*, **is** an inner product on V. The Minkowski metric tensor η is the inner product of Minkowski vector space

$$v \cdot u \equiv (v, u) \equiv \eta\,(v, u) = \eta_{\mu\nu} v^\mu u^\nu = -v^0 u^0 + v^1 u^1 + v^2 u^2 + v^3 u^3 \tag{5}$$

The norm squared of a vector is

$$||v||^2 = v \cdot v = \eta\,(v, v) = \eta_{\mu\nu} v^\mu v^\nu = -\left(v^0\right)^2 + \left(v^1\right)^2 + \left(v^2\right)^2 + \left(v^3\right)^2 \tag{6}$$

In Minkowski space there are three types of vectors, classified by their norm

$$\begin{aligned} &\eta\,(v, v) < 0 \quad \textit{timelike} \\ &\eta\,(v, v) = 0 \quad \textit{null / lightlike} \\ &\eta\,(v, v) > 0 \quad \textit{spacelike} \end{aligned} \tag{7}$$

The normalization of a timelike vector is to -1 and the normalization of a spacelike vector is to $+1$.

1.1.2 Raising and lowering indices

The non-degeneracy of the metric tensor (4) means that the matrix of $\eta_{\mu\nu}$ has non-zero determinant, no zero elements on the diagonal in the canonical form ($\eta_{\mu\nu} = diag\,(-1, 1, 1, 1)$), i.e., it has an inverse. The metric tensor η takes a vector and returns a covector, and the *inverse metric tensor* η^{-1} takes a covector and returns a vector. η^{-1} is a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$-tensor, denoted as $\eta^{\mu\nu}$, with the property

$$\eta^{\mu\rho} \eta_{\rho\nu} = \delta^\mu_\nu \tag{8}$$

$\eta^{\mu\nu}$ is the induced inner product on the dual space. The metric tensor provides a unique identification between vectors and covectors ("contravariant vectors" and "covariant vectors", "upper index vectors" and "lower index vectors").

The covector *dual* to the vector v^μ is

$$v_\mu \equiv \eta_{\mu\nu} v^\nu \tag{9}$$

Lowering an index with the metric

The contraction (summation) with either index of $\eta_{\mu\nu}$ is the same because the metric is symmetric. We say that we “lower the index” with the metric.

The vector *dual* to the covector ω_μ is

$$\omega^\mu \equiv \eta^{\mu\nu}\omega_\nu \tag{10}$$

Raising an index with the inverse metric

We say that we “raise the index” with the inverse metric.

The scalar product $v \cdot u$ can be written as

$$v_\nu u^\nu = v^\mu \eta_{\mu\nu} u^\nu = v^\mu u_\mu \tag{11}$$

where on the left we lowered the index of v and on the right we lowered the index of u. In abstract terms, the covector dual to v acting on the vector u, equals the covector dual to u acting on the vector v, equals the inner product between the vectors v and u. In matrix notation (11) reads

$$\begin{pmatrix} & v_\nu & \end{pmatrix}\begin{pmatrix} \\ u^\nu \\ \\ \end{pmatrix} = \begin{pmatrix} & v^\mu & \end{pmatrix}\begin{pmatrix} & & \\ & \eta_{\mu\nu} & \\ & & \end{pmatrix}\begin{pmatrix} \\ u^\nu \\ \\ \end{pmatrix} = \begin{pmatrix} & v^\mu & \end{pmatrix}\begin{pmatrix} \\ u_\mu \\ \\ \end{pmatrix} \tag{12}$$

1.1.3 Lorentz transformations preserve the inner product

Consider a linear operator on V

$$\begin{aligned} \Lambda : V &\to V \\ v &\to v' = \Lambda v \end{aligned} \tag{13}$$

Λ is called an *orthogonal operator* if it preserves the inner product between any two vectors v, u (thus also preserves the norm of any vector)

$$\eta\left(\Lambda v, \Lambda u\right) = \eta\left(v, u\right) \tag{14}$$

In components (14) reads

$$\eta_{\mu'\nu'}\left(\Lambda v\right)^{\mu'}\left(\Lambda u\right)^{\nu'} = \eta_{\mu'\nu'}\Lambda^{\mu'}{}_{\rho} v^\rho \Lambda^{\nu'}{}_{\sigma} u^\sigma = \eta_{\rho\sigma} v^\rho u^\sigma \tag{15}$$

(15) should be satisfied for any v and u , so we can write this condition on Λ as

$$\Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}\eta_{\mu'\nu'} = \eta_{\rho\sigma} \tag{16}$$

Preserving the inner product between any two vectors is the same as preserving the metric tensor. Such Λ is called a Lorentz transformation, and an object that under Lorentz transformation transforms as

$$v^{\mu'} = \Lambda^{\mu'}_{\ \nu} v^{\nu} \tag{17}$$

is called a 4-vector.

1.2 Kinematic 4-Vectors

1.2.1 The position 4-vector

The *position 4-vector* in Minkowski spacetime is

$$x^{\mu} = (t, \boldsymbol{x}) \tag{18}$$

In the rest frame of an observer $dx^i = 0$. The time measured by a clock carried by the observer is called *proper time*, and is denoted by τ. The invariant line element in this frame is

$$ds^2 = -d\tau^2 \tag{19}$$

Therefore the proper time $d\tau$ is an invariant. In an external observer frame

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{20}$$

Since $d\tau^2 = dt^2 \left(1 - \left(\frac{d\boldsymbol{x}}{dt}\right)^2\right)$, we have the time dilation conversion relation

$$\frac{dt}{d\tau} = \gamma \tag{21}$$

Lorentz factor conversion

The proper time lapsed between two events A, B is the positive length of the worldline between the events

$$\Delta\tau_{AB} = \int_{\tau(A)}^{\tau(B)} d\tau \tag{22}$$

Notice that the coordinate time difference $\Delta t_{AB} = t(A) - t(B)$ between two events A, B depends only on the **endpoints** of the worldline between the events, but it is **not invariant**. The proper time difference $\Delta\tau_{AB}$ on the other hand, **is an invariant**, with the "cost" of being dependent on the **path** of the worldline between the events, and not just the endpoints.

It is convenient to choose τ as a parameter for the worldline of a massive particle $x^\mu(\tau)$. It is an *arc-length parameter*. It means that we set $x^\mu(\tau = 0)$ at some point on the worldline, and then any other point $x^\mu(\tau)$ is specified as the point with distance τ along the worldline from $x^\mu(\tau = 0)$.

1.2.2 The velocity 4-vector

The *velocity 4-vector* of a massive particle is defined as

$$u^\mu := \frac{dx^\mu}{d\tau} \tag{23}$$

Since $d\tau$ is invariant, u^μ transforms like x^μ (as a 4-vector). Since we use an arc-length parameter τ, the 4-velocity is normalized to -1 (it is timelike)

$$\begin{aligned} u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d\boldsymbol{x}}{d\tau}\right)^2 \\ &= -\left(\frac{1}{d\tau}\right)^2 \left(dt^2 - d\boldsymbol{x}^2\right) = -1 \end{aligned} \tag{24}$$

where we used (20).

From (23) and (21) it follows (using the chain rule)

$$u^\mu = (\gamma, \gamma\boldsymbol{v}) \tag{25}$$

4-velocity from 3-velocity

In the rest frame of a massive particle $\gamma = 1, \boldsymbol{v} = \boldsymbol{0}$ and

$$u^\mu = (1, \boldsymbol{0}) \tag{26}$$

1.2.3 The momentum 4-vector

The *momentum 4-vector* is defined for massive particles as

$$p^\mu := m u^\mu \tag{27}$$

From (25) and (24) it follows that

$$p^\mu = (E, \boldsymbol{p}) = (\gamma m, \gamma m \boldsymbol{v}) \tag{28}$$

4-momentum from 3-velocity

$$p^\mu p_\mu = -m^2 \tag{29}$$

Kinematic Lorentz scalar

The mass is a Lorentz scalar (invariant), it classifies different particles.

We have

$$E = m\gamma \tag{30}$$

$$\boldsymbol{p} = \gamma m \boldsymbol{v} \tag{31}$$

and

$$E^2 = m^2 + \boldsymbol{p}^2 \tag{32}$$

In the rest frame of a massive particle $\gamma = 1, \boldsymbol{v} = \mathbf{0}$ and

$$p^\mu = (m, \mathbf{0}) \tag{33}$$

$$E = m \tag{34}$$

1.2.4 The wave 4-vector

For massless particles $ds^2 = d\tau^2 = 0$, so we need to use different parameter for the worldline. An affine parameter λ is a parameter such that the velocity 4-vector of a massless particle satisfies both

$$u^\mu u_\mu = 0 \tag{35}$$

(u^μ is null) and

$$\frac{du^\mu}{d\lambda} = 0 \tag{36}$$

(λ is affine). For example, light propagating in the positive x direction has 4-velocity

$$u^\mu = (1, 1, 0, 0) \tag{37}$$

The *momentum 4-vector* is defined for massless particles as

$$p^\mu = (E, \boldsymbol{p}) \tag{38}$$

such that

$$p^\mu p_\mu = 0 \tag{39}$$

So

$$E = |\boldsymbol{p}| \tag{40}$$

(39) and (40) agree with (29) and (32) as special cases for $m = 0$.

The *wave 4-vector* is defined for massless particles as

$$k^\mu := \frac{1}{\hbar} p^\mu = (\omega, \boldsymbol{k}) \tag{41}$$

The dispersion relation is

$$k^\mu k_\mu = 0 \tag{42}$$

$$\omega = |\boldsymbol{k}| \tag{43}$$

1.2.5 The acceleration 4-vector

The *acceleration 4-vector* is defined as

$$a^\mu := \frac{du^\mu}{d\tau} \tag{44}$$

Recall that since we work with arc-length parameter τ, a particle has a constant 4-speed $u^\mu u_\mu$. The acceleration of a particle with constant speed is perpendicular to its velocity.

$$a^\mu u_\mu = 0$$

Proof: If $u^2 = u \cdot u = const$, then $\frac{d}{d\tau}\left(u^2\right) = 2u \cdot \frac{du}{d\tau} = 2u \cdot a = 0$. The classic example for this is a particle moving along a circle with constant speed. It has only a radial acceleration, perpendicular to the velocity which is tangent to the circle. There is a hidden assumption in the proof, that the inner product itself does not change along the trajectory (independent of τ). It applies to the Minkowski metric which is constant anywhere.

Exercise

1. Express the 4-acceleration a^μ with the 3-velocity $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$ and 3-acceleration $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$.
2. Check that $a^\mu u_\mu = 0$.
3. Write down the 4-acceleration in the rest frame of the accelerating particle.

Solution:

1.

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{du^\mu}{dt}\frac{dt}{d\tau} = \gamma \frac{du^\mu}{dt} = \gamma \left(\frac{d}{dt}\gamma, \frac{d}{dt}(\gamma \boldsymbol{v})\right) = \left(\gamma \frac{d\gamma}{dt}, \gamma \frac{d\gamma}{dt}\boldsymbol{v} + \gamma^2 \frac{d\boldsymbol{v}}{dt}\right) \tag{45}$$

where we used (21) and (25). Calculate $\frac{d\gamma}{dt}$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1-\boldsymbol{v}^2\right)^{-\frac{1}{2}} = \left(1-\boldsymbol{v}^2\right)^{-\frac{3}{2}} \boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} = \gamma^3 \boldsymbol{v}\cdot\boldsymbol{a} \tag{46}$$

Plug into (45) we find

$$a^\mu = \left(\gamma^4 \boldsymbol{v}\cdot\boldsymbol{a}\ ,\ \gamma^4\left(\boldsymbol{v}\cdot\boldsymbol{a}\right)\boldsymbol{v} + \gamma^2\boldsymbol{a}\right) \tag{47}$$

4-acceleration from 3-velocity

2.

$$\begin{aligned} a^\mu u_\mu &= -\gamma^4\left(\boldsymbol{v}\cdot\boldsymbol{a}\right)\gamma + \left(\gamma^4\left(\boldsymbol{v}\cdot\boldsymbol{a}\right)\boldsymbol{v} + \gamma^2\boldsymbol{a}\right)\cdot\left(\gamma\boldsymbol{v}\right) \\ &= -\gamma^5\left(\boldsymbol{v}\cdot\boldsymbol{a}\right)\left(1-\boldsymbol{v}^2-\gamma^{-2}\right) = -\gamma^5\left(\boldsymbol{v}\cdot\boldsymbol{a}\right)\left(\gamma^{-2}-\gamma^{-2}\right) = 0 \end{aligned} \tag{48}$$

3. In the rest frame $\boldsymbol{v}=0$ and $\gamma=1$. (47) becomes

$$a^\mu = \left(0, \boldsymbol{a}\right) \tag{49}$$

2 Collision Exercises

2.1 Particles Annihilation/Creation

2.1.1 Annihilation

Two particles with equal masses $m_1 = m_2 = m$ collide head-on. Both have the same speed before the collision $v = \frac{3}{5}$. After the collision there is a particle with mass M at rest. Find M. Is it more/less/equal the sum of the two masses?

Solution:
We choose the x axis as the line of the collision. The 4-momenta of the particles before the collision are

$$\begin{aligned} p_1^\mu &= \left(\gamma m, \gamma m v, 0, 0\right) \\ p_2^\mu &= \left(\gamma m, -\gamma m v, 0, 0\right) \end{aligned} \tag{50}$$

The total 4-momentum before the collision is

$$p_i^\mu = p_1^\mu + p_2^\mu = \left(2\gamma m, 0, 0, 0\right) \tag{51}$$

The total 4-momentum after the collision is

$$p_f^\mu = (M, 0, 0, 0) \tag{52}$$

since $\boldsymbol{v}_f = \mathbf{0}$ and $\gamma_f = 1$.

4-momentum conservation is

$$p_i^\mu = p_f^\mu \tag{53}$$

The energy conservation $p_i^0 = p_f^0$ is

$$\boxed{M = 2m\gamma > 2m} \tag{54}$$

The final mass M is the total energy of the system. It contains both the total mass of the constituent particles and their total kinetic energy.

2.1.2 Creation

A particle at rest with mass M decays (splits) into two particles with equal mass m. Due to momentum conservation they have the same speed v. Find v.

As before, (54)

$$M = 2m\gamma \tag{55}$$

⇒

$$\gamma^{-2} = 1 - v^2 = \left(\frac{2m}{M}\right)^2 \tag{56}$$

⇒

$$v = \sqrt{1 - \left(\frac{2m}{M}\right)^2} \tag{57}$$

There is a solution only for $2m < M$.

2.2 Pion Decay

A pion at rest decays into a muon and neutrino. The neutrino is massless. What is the muon speed?

The total 4-momentum before the decay is the 4-momentum of the pion at rest

$$p_i^\alpha = p_\pi^\alpha = (m_\pi, \mathbf{0}) \tag{58}$$

The 4-momentum of the muon is

$$p_\mu^\alpha = (E_\mu, \boldsymbol{P}_\mu) \tag{59}$$

The neutrino is massless so $E_\nu = |\boldsymbol{P}_\nu|$. Its 4-momentum is

$$p_\nu^\alpha = (|\boldsymbol{P}_\nu|, \boldsymbol{P}_\nu) \tag{60}$$

The total 4-momentum after the decay is (59)+(60)

$$p_f^\alpha = p_\mu^\alpha + p_\nu^\alpha = (E_\mu + |\boldsymbol{P}_\nu|, \boldsymbol{P}_\mu + \boldsymbol{P}_\nu) \tag{61}$$

One way to solve is by direct energy-momentum conservation.

4-momentum conservation $p_i^\alpha = p_f^\alpha$

$$(m_\pi, \boldsymbol{0}) = (E_\mu + |\boldsymbol{P}_\nu|, \boldsymbol{P}_\mu + \boldsymbol{P}_\nu) \tag{62}$$

The momentum conservation $p_i^k = p_f^k$ implies $\boldsymbol{P}_\nu = -\boldsymbol{P}_\mu$. The energy conservation $p_i^0 = p_f^0$ then implies

$$m_\pi = E_\mu + |\boldsymbol{P}_\mu| = \sqrt{m_\mu^2 + |\boldsymbol{P}_\mu|^2} + |\boldsymbol{P}_\mu| \tag{63}$$

$\Rightarrow$

$$(m_\pi - |\boldsymbol{P}_\mu|)^2 = m_\mu^2 + |\boldsymbol{P}_\mu|^2 \tag{64}$$

$\Rightarrow$

$$m_\pi^2 - 2m_\pi |\boldsymbol{P}_\mu| = m_\mu^2 \tag{65}$$

$\Rightarrow$

$$\frac{m_\pi^2 - m_\mu^2}{2m_\pi} = |\boldsymbol{P}_\mu| = m_\mu \gamma |\boldsymbol{v}| = \frac{m_\mu |\boldsymbol{v}|}{\sqrt{1 - \boldsymbol{v}^2}} \tag{66}$$

square the equation, and more algebra yields the solution

$$v = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \tag{67}$$

A second way to solve is by the conservation of $p^\alpha p_\alpha$ (since p^α is conserved). From (58)

$$p_i^\alpha p_{i\alpha} = -m_\pi^2 \tag{68}$$

from (61) (and momentum conservation $\boldsymbol{P}_\mu + \boldsymbol{P}_\nu = \boldsymbol{0}$)

$$p_f^\alpha p_{f\alpha} = -\left(E_\mu + |\boldsymbol{P}_\mu|\right)^2 = -\left(\gamma m_\mu + \gamma m_\mu \left|\boldsymbol{v}\right|\right)^2 = -m_\mu^2 \frac{\left(1+|\boldsymbol{v}|\right)^2}{1-\boldsymbol{v}^2} \tag{69}$$

Since $p_i^\alpha p_{i\alpha} = p_f^\alpha p_{f\alpha}$

$$m_\pi^2 = m_\mu^2 \frac{\left(1+|\boldsymbol{v}|\right)^2}{1-\boldsymbol{v}^2} \tag{70}$$

Again, algebra yields (67).

2.3 Scattering Of Protons

A proton with energy E collides with another proton at rest. The outcome is four protons. What is the minimal energy E (as a function of the mass of a proton m), for that scattering process to occur?

We will use the fact that $p_\mu p^\mu$ is both invariant under change of reference frame, and conserved with time.

Before the collision, we write the total 4-momentum in the lab frame

$$p^\mu_{lab,i} = (E+m, \boldsymbol{P}) \tag{71}$$

where E and $\boldsymbol{P}$ are the energy and momentum of the moving proton, with $E^2 = m^2 + \boldsymbol{P}^2$. The proton at rest has only a rest energy m.

After the collision, we use the center of mass frame. The sum of the momenta of the four protons is zero in center of mass frame (just as for the two protons before the collision). Still, each proton can have some velocity $\boldsymbol{v}$ relative to the center of mass. We look for minimal energy condition, so we want no energy waste on new kinetic energy for the new protons (except for the center of mass to have the same momentum as before the scattering). It means that in the c.o.m frame we take $\boldsymbol{v} = \boldsymbol{0}$ for all the four protons. Therefore $\gamma = 1$, and each proton contributes to the energy in the c.o.m frame an amount of $E = \gamma m = m$. The final total 4-momentum in the c.o.m frame is

$$p^\mu_{com,f} = (4m, \boldsymbol{0}) \tag{72}$$

Now, since $p_\mu p^\mu$ is both invariant under change of reference frame, and conserved with time, we can compare

$$\left(p_{lab,i}\right)^\mu \left(p_{lab,i}\right)_\mu = \left(p_{com,f}\right)^\mu \left(p_{com,f}\right)_\mu \tag{73}$$

$\Rightarrow$

$$-\left(E+m\right)^2+\boldsymbol{P}^2=-\left(4m\right)^2 \tag{74}$$

plug $\boldsymbol{P}^2=E^2-m^2$

$$-\cancel{E^2}-2mE-m^2+\cancel{E^2}-m^2=-16m^2 \tag{75}$$

$\Rightarrow$

$$\boxed{E=7m} \tag{76}$$