Intermediate Asymptotic

In the previous lectures, we saw that the central limit theorem (CLT) is valid in the limit $N \to \infty$. In reality, N (or equivalently the measurement time of a process), might be large (long compared with some (process specific) time scale), however, it is never infinitely large (long). Therefore, it is interesting to investigate the precise meaning of large N. Consider the scaled sum of random variables,

$$Z = \frac{1}{\sigma\sqrt{N}} \sum_{i=1}^{N} x_i, \tag{1}$$

where the x_is are i.i.d. (independent, identically distributed) random variables. Assume that the fourth moment of x_i diverges, $\langle x_i^4 \rangle = \infty$; however, the variance σ^2 (σ is the standard deviation of x_i) is finite. Clearly, for any finite N, $\langle Z^4 \rangle = \infty$. On the other hand, if we assume that for finite, but large, N the distribution of Z is well approximated by a Gaussian distribution then we have a finite $\langle Z^4 \rangle$. This contradiction stems from the fact that the central limit theorem does not work well in the tails of the PDF (probability density function) of Z.

We are interested in the large N limit of the PDF of Z, $p_z(Z)$. In particular, we are interested in the corrections to the Gaussian behavior. Let $\widetilde{p}_x(k)$ be the Fourier transform of $p_x(x)$ and $\widetilde{p}_z(k)$ the Fourier transform of $p_z(Z)$. Using the i.i.d. property of the x_is and the convolution theorem we may write (if you don't remember the derivation, please see the lecture notes for the CLT)

$$\widetilde{p}_z(k) = \left[\widetilde{p}_x\left(\frac{k}{\sigma\sqrt{N}}\right)\right]^N. \tag{2}$$

(Obviously, inversion of the expression above yields the exact PDF $p_z(Z)$ for any finite N.) We rewrite the expression above as,

$$\widetilde{p}_z(k) = e^{N \ln\left[\widetilde{p}_x\left(\frac{k}{\sigma\sqrt{N}}\right)\right]}. \tag{3}$$

The Taylor expansion of $\ln\left[\widetilde{p}_x\left(\frac{k}{\sigma\sqrt{N}}\right)\right]$,

$$\ln\left[\widetilde{p}_x\left(\frac{k}{\sigma\sqrt{N}}\right)\right] = \sum_{l=1}^{\infty} \frac{\left(-i\frac{k}{\sigma\sqrt{N}}\right)^l}{l!} c_l, \tag{4}$$

is called the cumulants expansion, the $\{c_l\}\,s$ are the cumulants. We assume here that all the cumulants are finite; namely, all the moments are finite. Note that the $l'th$ cumulant of the un-normalized sum ($Y = \sigma\sqrt{N}Z$) is Nc_l, N times the cumulant of x, so in this sense cumulants are additive. For simplicity, we will use the following settings $\langle x^{2n+1} \rangle = 0$ (n is a positive integer), $\langle x^2 \rangle = \sigma^2$ and $\langle x^4 \rangle = m_4$. Using these settings we may expand the characteristic function as

$$\widetilde{p}_x(k) = 1 - \frac{\sigma^2}{2}k^2 + \frac{m_4}{4!}k^4 + O(k^6). \tag{5}$$

Using the expansion $\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + O(\varepsilon^3)$ we write,

$$\ln\left[\widetilde{p}_x(k)\right] = -\frac{\sigma^2}{2}k^2 + \frac{(m_4 - 3\sigma^4)}{4!}k^4 + O(k^6), \tag{6}$$

(note that for our case, the fourth cumulant is $c_4 = (m_4 - 3\sigma^4)$).

Inverting the Fourier transform, $\widetilde{p}_z(k)$, we write

$$p_z(Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikZ} e^{-N\left(\frac{k^2}{2N} - \frac{c_4}{4!\sigma^4 N^2}k^4 + O\left(\frac{k^6}{N^3}\right)\right)} dk. \tag{7}$$

We are interested in $N >> 1$ and therefore, we expand $e^{\left(\frac{c_4}{4!N\sigma^4}k^4+O(\frac{k^6}{N^2})\right)}$ as

$$e^{\left(\frac{c_4}{4!N\sigma^4}k^4+O(\frac{k^6}{N^2})\right)} = 1 + \frac{\kappa}{4!N}k^4 + O(\frac{k^6}{N^2}), \tag{8}$$

(we defined here the kurtosis $\kappa \equiv \frac{c_4}{\sigma^4}$) which leads to,

$$\begin{aligned} p_z\left(Z\right) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ikZ-\frac{k^2}{2}}\left(1+\frac{\kappa}{4!N}k^4+O(\frac{k^6}{N^2})\right)dk \\ &= \frac{1}{2\pi}\left(1+\frac{\kappa}{4!N}\frac{\partial^4}{\partial Z^4}+...\right)\int\limits_{-\infty}^{\infty} e^{-iqZ-\frac{q^2}{2}}dq. \end{aligned} \tag{9}$$

Integrating and carrying out the derivative one obtains,

$$p_z\left(Z\right) = \frac{1}{\sqrt{2\pi}}e^{-\xi^2}\left(1+\frac{\kappa}{4!N}\left[3-12\xi^2+4\xi^4\right]+O\left(\frac{1}{N^2}\right)\right), \tag{10}$$

where we defined

$$\begin{aligned} \xi &= Z/\sqrt{2}, \\ \kappa &= \frac{c_4}{\sigma^4}. \end{aligned} \tag{11}$$

The first term in equation (10) is a Gaussian term, an obvious manifestation of the CLT. The second term is a correction term which should be small for the whole approach to be valid.

One can easily see that for $\xi \to \infty$ the second term is not small. Hence, the Gaussian approximation fails. In other words, for large $|Z|$ (the tails of the distribution) we do not expect the CLT to work. To get a rough approximation where the Gaussian distribution fails we define ξ_{edge} and the corresponding Z_{edge} in such a way that,

$$1 = \frac{\kappa}{4!N}4\xi_{edge}^4, \tag{12}$$

(this definition is obvious from equation (10)). A simple transformation yields,

$$Z_{edge} = \frac{(4!)^{1/4}}{\kappa^{1/4}}N^{1/4}. \tag{13}$$

The Gaussian regime grows as $N^{1/4}$. Note that $\kappa = 0$ when $p_x\left(x\right)$ is Gaussian and then $Z_{edge} = \infty$, meaning that the Gaussian behavior holds everywhere. This is a manifestation of the fact that the PDF of the sum of Gaussian i.i.d. random variables is also Gaussian for any N.

Lorentz or Cauchy PDF

Not all PDFs have a finite variance. A counter example is the Cauchy or Lorentz PDF

$$p\left(x\right) = \frac{1}{\pi}\frac{a}{x^2+a^2}. \tag{B1}$$

The characteristic function of this PDF is given by,

$$\widetilde{p}\left(k\right) = \int\limits_{-\infty}^{\infty}\frac{1}{\pi}\frac{a}{x^2+a^2}e^{-ikx}dx = e^{-a|k|}. \tag{B2}$$

The integral can be calculated in the complex plain. For negative k one has to consider the following closed contour:

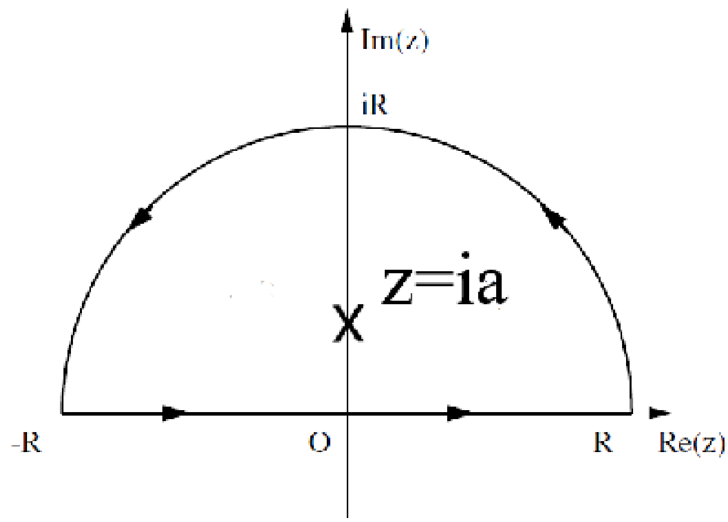

Figure 1: The integration contour for negative values of k in the calculation of the characteristic function of the Lorentz (Cauchy) distribution.

The integral vanishes along the arc due to the fact that the imaginary part of z is positive while k is negative and the result is:

$$\widetilde{p}(k<0)=\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{a}{x^2+a^2}e^{i|k|x}dx=2\pi i\mathrm{Res}\left(\frac{1}{\pi}\frac{a}{x^2+a^2}e^{i|k|x},x=ia\right)=2\pi i\frac{1}{\pi}\frac{a}{2ia}e^{-a|k|}=e^{-a|k|}.$$

For $k>0$ one has to use the closed integral over the contour closed throught the negative imaginary side and the result obtained is the same.

The first moment vanishes due to the symmetric nature of the PDF. However, the second moment diverges. Both properties can be seen by differentiating the characteristic function at $k=0$.

$$\frac{d}{dk}\widetilde{p}(k)\,|_{k=0} = \frac{d}{dk}e^{-a|k|}|_{k=0}=a\left(e^{ak}\Theta(-k)-e^{-ak}\Theta(k)\right)|_{k=0}=0 \tag{B3}$$

$$\begin{aligned}-\frac{d^2}{dk^2}\widetilde{p}(k)\,|_{k=0} &= -\frac{d}{dk}a\left(e^{ak}\Theta(-k)-e^{-ak}\Theta(k)\right)|_{k=0} \\ &= \left[-a^2\left(e^{ak}\Theta(-k)+e^{-ak}\Theta(k)\right)+a\left(e^{-ak}+e^{ak}\right)\delta(k)\right]|_{k=0}=\infty\end{aligned} \tag{B4}$$

One-Sided PDFs

Consider the random variable $0<y<\infty$ whose PDF is $p(y)$. Such PDFs are called one sided. The Fourier transform is irrelevant for those PDFs (the integral in the Fourier transform is from $-\infty$ to ∞) and the more useful transform is the Laplace transform, which is defined as,

$$\widetilde{p}(s)=\int_{0}^{\infty}e^{-sy}p(y)\,dy. \tag{C1}$$

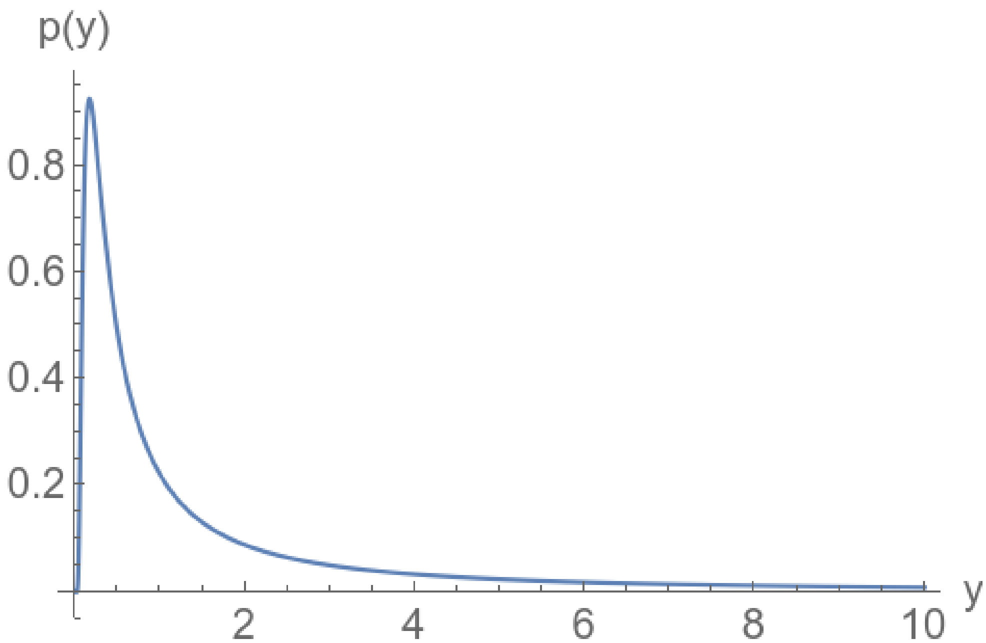

Figure 2: Smirnov's probability density function.

Note that this transform is also a moment generating function since,

$$\widetilde{p}(s) = \int_0^\infty \sum_{n=0}^\infty \frac{(-sy)^n}{n!} p(y)\, dy = \sum_{n=0}^\infty \frac{(-s)^n}{n!} \langle y^n \rangle . \tag{C2}$$

Hence,

$$\langle y^n \rangle = (-1)^n \frac{d^n}{ds^n} \widetilde{p}(s) \,|_{s=0}. \tag{C3}$$

Obviously, the above analytical expressions are only valid if all the moments are finite.

In general, the Laplace transfrom of a PDF can be expanded around $s = 0$ as,

$$\widetilde{p}(s) = 1 - Bu^\alpha + O\left(u^{\min(2\alpha,1)}\right),$$

where, $B > 0$ and $0 < \alpha < 1$.
Consider for example the Smirnov PDF,

$$p(y) = \frac{1}{y\sqrt{4\pi y}} e^{-\frac{1}{4y}}, \tag{C4}$$

whose Laplace transform is

$$\widetilde{p}(s) = e^{-\sqrt{s}}. \tag{C5}$$

**********************Calculation of the Laplace transform***

$$\widetilde{p}(s) = \int_0^\infty e^{-sy} \frac{1}{y\sqrt{4\pi y}} e^{-\frac{1}{4y}} dy$$

Changing variables according to:

$$
\begin{aligned}
u &= \frac{1}{2\sqrt{y}}; du = -\frac{1}{4}y^{-3/2}dy \\
\widetilde{p}(s) &= \frac{2}{\sqrt{\pi}}\int\limits_0^\infty e^{-\frac{s}{4u^2}}e^{-u^2}du = \frac{2}{\sqrt{\pi}}\left(\frac{s}{4}\right)^{1/4}\int\limits_0^\infty e^{-\left(\frac{s}{4}\right)^{1/2}\left(\frac{1}{x^2}+x^2\right)}dx
\end{aligned}
$$

Changing variables again according to:

$$
\begin{aligned}
x &= u\left(\frac{4}{s}\right)^{1/4}; dx = du\left(\frac{4}{s}\right)^{1/4} \\
\widetilde{p}(s) &= \frac{2}{\sqrt{\pi}}\left(\frac{s}{4}\right)^{1/4}\left(\int\limits_0^1 e^{-\left(\frac{s}{4}\right)^{1/2}\left(\frac{1}{x^2}+x^2\right)}dx + \int\limits_1^\infty e^{-\left(\frac{s}{4}\right)^{1/2}\left(\frac{1}{x^2}+x^2\right)}dx\right)
\end{aligned}
$$

In the first integral we change variables according to $z = 1/x$ to obtain

$$
\begin{aligned}
\widetilde{p}(s) &= \frac{2}{\sqrt{\pi}}\left(\frac{s}{4}\right)^{1/4}\left(\int\limits_1^\infty \frac{1}{z^2}e^{-\left(\frac{s}{4}\right)^{1/2}\left(\frac{1}{z^2}+z^2\right)}dz + \int\limits_1^\infty e^{-\left(\frac{s}{4}\right)^{1/2}\left(\frac{1}{x^2}+x^2\right)}dx\right) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{s}{4}\right)^{1/4}\left(\int\limits_1^\infty \left(\frac{1}{x^2}+1\right)e^{-\left(\frac{s}{4}\right)^{1/2}\left(\frac{1}{x^2}+x^2\right)}dx\right)
\end{aligned}
$$

Changing variables again

$$
\begin{aligned}
w &= x - 1/x; dw = \left(1 + 1/x^2\right)dx; \frac{1}{x^2} + x^2 = (x - 1/x)^2 + 2; \\
\widetilde{p}(s) &= \frac{2}{\sqrt{\pi}}\left(\frac{s}{4}\right)^{1/4}\left(\int\limits_0^\infty e^{-\left(\frac{s}{4}\right)^{1/2}\left(w^2+2\right)}dw\right) = e^{-\sqrt{s}}\frac{2}{\sqrt{\pi}}\left(\frac{s}{4}\right)^{1/4}\left(\int\limits_0^\infty e^{-\left(\frac{s}{4}\right)^{1/2}w^2}dw\right)
\end{aligned}
$$

And changing variables one last time yields:

$$
\begin{aligned}
q &= w\left(\frac{s}{4}\right)^{1/4}; dq = dw\left(\frac{s}{4}\right)^{1/4}; \\
\widetilde{p}(s) &= e^{-\sqrt{s}}\frac{2}{\sqrt{\pi}}\left(\int\limits_0^\infty e^{-q^2}dq\right) = e^{-\sqrt{s}}.
\end{aligned}
$$

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In this case, the first moment diverges; and therefore, the Laplace transform is non-analytical in the vicinity of $s \to 0$.

$$
\widetilde{p}(s)\,\tilde{}\,1 - s^{1/2} + O(s).
$$

Power-Law PDFs

Consider the generic PDF of the form,

$$
p(t) \sim At^{-(1+\alpha)}. \tag{D1}
$$

with $0 < \alpha < 1$. The integer moments diverge in this case, i. e., $\langle t \rangle = \infty$. In order to study the behavior of the Laplace transform, in the vicinity of $s \to 0$, of such a PDF we express it as:

$$\widetilde{p}(s) = \int_0^{t_0} p(t) e^{-st} dt + A \int_{t_0}^{\infty} t^{-(1+\alpha)} dt + A \int_{t_0}^{\infty} t^{-(1+\alpha)} \left(e^{-st} - 1\right) dt + \varepsilon \tag{D2}$$

Here, t_0 is large in the sense that the large t approximation of $p(t)$ is valid for $t > t_0$. Under this condition ε is small, more precisely, $\varepsilon = 0$ if $p(t) = At^{-(1+\alpha)}$ for $t > t_0$. In the limit of $s \to 0$, the first two terms in the RHS (right hand side) of equation (D2) are equal to one (from the normalization condition). In addition, the small s expansion of these two terms is of the form $C + Bs$ (C and B are constants in s).

The third term may be written as:

$$A \int_{t_0}^{\infty} t^{-(1+\alpha)} \left(e^{-st} - 1\right) dt = As^{\alpha} \int_{st_0}^{\infty} x^{-(1+\alpha)} \left(e^{-x} - 1\right) dx. \tag{D3}$$

If we consider the limit $st_0 \to 0$ (namely $s << 1/t_0$) it becomes,

$$As^{\alpha}\Gamma(-\alpha), \tag{D4}$$

where $\Gamma(x)$ is the gamma function. Collecting all the terms we find that

$$\widetilde{p}(s) = 1 + A\Gamma(-\alpha) s^{\alpha} + ... \tag{D5}$$

(note that $\Gamma(-\alpha) < 0$ for $0 < \alpha < 1$ and ... refers to terms of higher order in s). This kind of relations, between the long time behavior and the small Laplace conjugate one, are known as the Tauberian and Abelian relations.