QFT Home Assignment # 1. Submission date 21.11.2021

1. Peskin and Schroeder: Problem 2.1.

2. Consider the scalar field theory in curved space with the Lagrangian:

$$L = \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\mu \phi - V(\phi) \tag{1}$$

and the action

$$S = \int d^4x \sqrt{-g}\, L \tag{2}$$

where g is the determinant of the metric.
Show that

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} |_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{3}$$

with $\eta_{\mu\nu} = \{1, -1, -1, -1\}$ being the Minkowski metric, gives the expression for the energy momentum tensor $T^{\mu\nu}$ in flat space. In general the most efficient way to compute energy momentum tensor for a field theory in flat space is to embed the theory into curved background, differentiate with respect to the metric as in (3), and then set the metric back to flat.

3. Verify that the Lagrangian density

$$L = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2}{2} \phi_a \phi_a \tag{4}$$

for a triplet of real fields ϕ_a (a=1,2,3) is invariant under infinitesimal $SO(3)$ rotation by θ

$$\phi_a \to \phi_a + \theta\, \epsilon_{abc} \phi_b\, n_c$$

where n_c is a unit vector. Compute the Noether current. Deduce that

$$Q_a = \int d^3x\, \epsilon_{abc}\, \dot{\phi}_b\, \phi_c$$

are conserved charges and verify this directly using equations of motion.

4. Consider the theory of a single charged scalar field with the Lagrangian

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi^* - \frac{m^2}{2} \phi^* \phi - \frac{\lambda}{4!} (\phi^* \phi)^2 \tag{5}$$

a). Quantize the theory using Hamiltonian formalism - find conjugate momenta, commutation relations, and write down the Hamiltonian.

b). Solve the free limit ($\lambda = 0$). Find the representation of the fields in terms of creation and annihilation operators, find the eigenstates of the Hamiltonian and the spectrum. Calculate (in momentum space) the Feynman propagator, defined as $\langle 0|\phi(x)\, \phi^*(y)|0\rangle$. Show that

$$\langle 0|\phi(x)\, \phi(y)|0\rangle = \langle 0|\phi^*(x)\, \phi^*(y)|0\rangle = 0$$

c). For the interacting theory, develop the perturbation theory in terms of Feynmann diagrams and describe Feynmann rules.

5. The Lagrangian density for a two-component complex scalar field

$$\vec{\phi} \; = \; \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

is given by

$$L \, = \, \partial_\mu \vec{\phi}^\dagger \, \partial^\mu \vec{\phi} \; - \; m^2 \vec{\phi}^\dagger \, \vec{\phi}$$

where Hermitean conjugation is defined by

$$\vec{\phi}^\dagger = [\phi_1^*, \, \phi_2^*]$$

(a) Show that the above Lagrangian is invariant under the following global SU(2) symmetry

$$\phi_i \; \rightarrow \phi_i' \; = \; \left(exp \left\{ \frac{i \, \vec{\alpha} \vec{\sigma}}{2} \right\} \right) \; \phi_j$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ an arbitrary coordinate-independent vector, $\vec{\sigma}$ are the Pauli matrices, and i, j = 1, 2.

(b) Find the conserved current j_μ^a and charge Q^a corresponding to this symmetry (here a = 1, 2, 3).

6. Peskin and Schroeder: Read "Particle creation by a Classical Source" (Chapter 2, page 32-33). Do problem 4.1.