Levy's Generalized CLT

Let $\{x_1, x_2, ...x_N\}$ be a set of N independent, identically and symmetrically ($p_{x_i}(x_i) = p_{x_i}(-x_i)$ and hence $\langle x_i \rangle = 0$) distributed random variables. The variance of each variable diverges, namely, $\langle x_i^2 \rangle = \infty$. As an example consider the Lorentz distribution (see Lecture 3 notes). The CLT does not hold for this case due to the diverging variance. However, a generalized form of this theorem, known as Levy's generalized CLT is valid. Let's define,

$$Z = \frac{\sum_{i=1}^{N} x_i}{N^\beta}, \tag{E1}$$

the proper choice of β will be clarified in what follows.

The PDF of x_i is assumed to have the following asymptotics as $x_i \to \infty$

$$p_{x_i}(x_i) \sim A|x_i|^{-(1+\mu)}, \tag{E2}$$

where $0 < \mu < 2$ (hence, the variance diverges).

In Fourier space, and in the vicinity of $k \to 0$,

$$\widetilde{p}_{x_i}(k) \sim 1 - \widetilde{A}|k|^\mu \tag{E3}$$

where $\widetilde{A} = \frac{A}{\frac{\mu}{\pi}\Gamma(\mu)\sin\left(\frac{\pi\mu}{2}\right)} = \frac{2A\Gamma(1-\mu)}{\alpha}\cos\left(\frac{\pi\mu}{2}\right)$. Using the convolution theorem, in a similar fashion to what we did in deriving the CLT, we write

$$\widetilde{p}_Z(k) = \left(\widetilde{p}_{x_i}\left(\frac{k}{N^\beta}\right)\right)^N = \left(1 - \frac{\widetilde{A}|k|^\mu}{N^{\beta\mu}} + ...\right)^N, \tag{E4}$$

(... are terms of higher order in k). If we set, $\beta\mu = 1 \to \beta = 1/\mu$, we obtain in the limit of $N \to \infty$,

$$\widetilde{p}_Z(k) \sim e^{-\widetilde{A}|k|^\mu} \tag{E5}$$

The Fourier transform of the PDF of Z is a streched exponent. Inverting the Fourier transform yields the PDF of Z. These functions are known as Levy distributions or stable Levy distributions and are denoted by $l_{\mu,\widetilde{A},0}(Z)$ (the first index denotes the value of the exponent, μ; the second index denotes the value of the coefficient, $\widetilde{A}$, and the third index takes the values 0 or 1 which correspond to the symmetric stable Levy distribution and the one sided Levy distribution, respectively). In the case considered above, the PDF is obviously symmetric due to the symmetric nature of $p_{x_i}(x_i)$. μ, is called the characteristic exponent. In the special case of $\mu = 2$ we recover the CLT. For the case of $\mu = 1$ we obtain the Lorentzian PDF for which $\widetilde{p}_Z(k) \sim e^{-\widetilde{A}|k|}$. For other values of μ, there is no close form of $l_{\mu,\widetilde{A},0}(Z)$. It is possible to express $l_{\mu,\widetilde{A},0}(Z)$ in terms of Fox functions. In addition, it is always possible to numerically invert the Fourier transform.

The law of large numbers

Consider the scaled sum of N independent identically distributed (i.i.d.) random variables $\{t_1, t_2, ...t_N\}$ each characterized by the PDF $p_t(t_i)$ and $t_i > 0$,

$$Z_N = \frac{1}{N^\beta}\sum_{i=1}^{N} t_i \tag{1}$$

Let $\widetilde{p}_t(u)$ be the Laplace transform of $p_t(t_i)$. When $\beta = 1$, Z is a random variable, however, we expect that in the limit of large N,

$$\lim_{N\to\infty} Z_N = \langle t \rangle = \int_0^\infty t_i p_t(t_i)\, dt_i, \tag{2}$$

provided that $\langle t \rangle$ is finite. Using the i.i.d. property and the convolution theorem we may write,

$$\widetilde{p}_{Z_N}(u) = \left(\widetilde{p}_t\left(\frac{u}{N^{\beta}}\right)\right)^N . \tag{3}$$

The small u expansion of $\widetilde{p}_t(u)$ is of the form:

$$\widetilde{p}_t(u) \sim 1 - Au^{\alpha} \qquad\qquad 0 < \alpha \leq 1. \tag{4}$$

For $\alpha = 1$, the first moment is finite and $A = \langle t \rangle$. For $\alpha < 1$ the first moment diverges. Substituting the expansion in the Laplace transform, $\widetilde{p}_Z(u)$, one obtains:

$$\widetilde{p}_{Z_N}(u) = \left(1 - A\frac{u^{\alpha}}{N^{\alpha\beta}}\right)^N . \tag{5}$$

If we set $\beta = 1/\alpha$ we find

$$\lim_{N\to\infty} \widetilde{p}_{Z_N}(u) = e^{-Au^{\alpha}}. \tag{6}$$

If $\alpha = 1, \beta = 1$

$$\lim_{N\to\infty} p_{Z_N}(Z_N) = \delta(Z_N - A) = \delta(Z_N - \langle t \rangle). \tag{7}$$

The inverse Laplace transform of $e^{-Au^{\alpha}}$ (for $0 < \alpha < 1$) is denoted by $l_{\alpha,A,1}(Z_N)$, hence,

$$\lim_{N\to\infty} p_{Z_N}(Z_N) = l_{\alpha,A,1}(Z_N) \tag{8}$$

and

$$\int_0^{\infty} e^{-ut} l_{\alpha,A,1}(t) = e^{-Au^{\alpha}}. \tag{9}$$

The PDF $l_{\alpha,A,1}(t)$ is called the one-sided Levy stable distribution. Note, that the "average" (the scaled sum) remains random (its PDF is not a delta function) even in the limit $N \to \infty$.

Generating Levy flights on a computer

In order to get better insights into power-law statistics, it is recommended to perform some numerical experiments. It is easy to generate a random number, x, uniformly distributed in the interval $[0, 1]$. Starting with such a number we can find a transformation yielding a random number characterized by a power-law statistics. One such transformation is:

$$t = (1 - x)^{-1/\alpha}. \tag{10}$$

x close to 1 yields a large t and x close to 0 yields t close to 1. So, $1 < t < \infty$. Using the chain rule,

$$p_t(t) = p_x(x)\left|\frac{dx}{dt}\right|, \tag{11}$$

$$\frac{dt}{dx} = \frac{1}{\alpha}(1 - x)^{-1/\alpha - 1} = \frac{1}{\alpha}(1 - x)^{-\frac{1}{\alpha}(1+\alpha)} = \frac{1}{\alpha}t^{(1+\alpha)} \tag{12}$$

and $p_x(x) = 1$ (it is uniformly distributed in the interval $[0, 1]$) we find

$$p_t(t) = \alpha t^{-(1+\alpha)} \qquad\qquad t > 1. \tag{13}$$

Construct in the computer, Levy flights and Pearson walks, in 2D. This can be done in the following way. Start in the origin, and choose a direction, $0 < \theta < 2\pi$, from a uniform distribution. Then, draw the jump length from a PDF $f(r)$ ($0 < r < \infty$). If $f(r)$ is light tailed, e.g., exponentially distributed, the walk is called a Pearson walk. If $f(r)$ is fat tailed, the walk is called a Levy flight.

Generate in the computer Levy flights for 3 different values of α (between 0 and 1). For each value, generate many trajectories and determine the mean squared displacement as a function of the number of steps. Quantify the dependence of the variance (and the ratio between the standard deviation and the mean) on the number of trajectories.

Repeat the same simulations for the Pearson walk with an exponential distribution of the jump lengths, $f(r) = \xi e^{-\xi r}$ and three different values of ξ.

Long Range Interacting Systems and Levy Statistics

Consider an infinite system of particles, interacting via a two body central field. The particles are assumed to be uniformly distributed in space, with a constant density. In the thermodynamic limit, $N \to \infty$, $V \to \infty$ and $N/V = \rho$. The force that the particles exert on a particle located on the origin is:

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) . \tag{14}$$

In what follows, we will consider the distribution of forces along the z direction for the Newtonian attraction in 3D. The force in the z direction exerted on a particle with mass m surrounded by particles of mass M is:

$$F_z = \sum_{i=1}^{N} \frac{GmM \cos\theta_i}{|\mathbf{r}_i|^2} . \tag{15}$$

The factor $\cos\theta_i$ yields the projection of the force on the z axis. G is Newton's gravitational constant and $\mathbf{r}_i$ is the vector connecting the particle of interest to the $i'th$ particle. The masses are randomly and uniformly distributed in space. We wish to find the characteristic function of F_z. Using the fact that the positions of the bath particles are statistically uncorrelated we find,

$$\left\langle e^{ikF_z} \right\rangle = \left[1 + \int_V d^3r \frac{e^{iGmMk\cos\theta/r^2} - 1}{V} \right]^N . \tag{16}$$

We used the assumption of a uniform distribution with a probability density $1/V$. We also added and subtracted 1 in preparation for the thermodynamic limit. Using $N = \rho V \to \infty$ we write,

$$\left\langle e^{ikF_z} \right\rangle = \left[1 + \frac{N}{V} \int_V \frac{e^{iGmMk\cos\theta/r^2} - 1}{N} d^3r \right]^N = \exp\left[-\rho \int_V d^3r \left(1 - e^{iGmMk\cos\theta/r^2} \right) \right] . \tag{17}$$

The above equation was derived by the relation,

$$\lim_{n\to\infty} (1 + x/n)^n = e^x .$$

The integral over the imaginary part vanishes, due to the symmetry of the problem.

$$i\rho 2\pi \int_0^\infty dr r^2 \int_{-1}^{1} du \left(\sin(GmMku/r^2) \right) = 0.$$

The integral to solve is

$$\left\langle e^{ikF_z} \right\rangle = \exp\left[-\rho 2\pi \int_0^\infty dr r^2 \int_{-1}^{1} du \left(1 - \cos\left(GmMku/r^2 \right) \right) \right] , \tag{18}$$

(here we defined $\cos\theta = u$). The symmetry allows us to write:

$$\left\langle e^{ikF_z} \right\rangle = \exp\left[-\rho 4\pi \int_0^\infty drr^2 \int_0^1 du \left(1-\cos\left(GmMku/r^2\right)\right)\right]. \quad (19)$$

Changing variables according to $GmMku/r^2 = 1/y^2$ and then integrating over u we find,

$$\begin{aligned} r &= \sqrt{GmM|k|u}y \\ dr &= \sqrt{GmM|k|u}dy \end{aligned}$$

$$\left\langle e^{ikF_z} \right\rangle = \exp\left[-\rho 4\pi \int_0^\infty dy \left(GmM|k|\right)^{3/2} y^2 \left(1-\cos\left(1/y^2\right)\right) \frac{2}{5}\right], \quad (20)$$

$$\left\langle e^{ikF_z} \right\rangle = \exp\left[-\frac{\rho 8\pi}{5} \left(GmM\right)^{3/2} |k|^{3/2} \int_0^\infty dyy^2 \left(1-\cos\left(1/y^2\right)\right)\right]. \quad (21)$$

Using the tables we obtain the integral on the RHS as $\sqrt{2\pi}/3$, enabling us to write the characteristic function as,

$$\left\langle e^{ikF_z} \right\rangle = \exp\left[-\frac{\rho\sqrt{28}}{15} \left(GmM\pi\right)^{3/2} |k|^{3/2}\right]. \quad (22)$$

The PDF of F_z is therefore, a symmetric Levy PDF with a characteristic exponent $\mu = 3/2$, $\quad l_{3/2,A,0}\left(F_z\right)$. $A = \frac{\rho\sqrt{28}}{15}\left(GmM\pi\right)^{3/2}$. Hence, the variance of the random force the stellar object experiences is infinite.

First passage time on 1D infinite space

Consider a particle, initially $(t = 0)$ located on $x_0 > 0$, diffusing along an infinite 1D line. We know that the Green function for free diffusion on a line is:

$$G\left(x, x', t\right) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{\left(x-x'\right)^2}{4Dt}}. \quad (23)$$

In order to calculate its first arrival at $x = 0$, we need to set an absorbing boundary there (this way a particle which arrives at that point doesn't continue the diffusion; therefore, each arrival is the first arrival). To satisfy this requirement we need to modify the Green function. The absorbing boundary condition is easily achieved by setting

$$G_a\left(x, x', t\right) = \frac{e^{-\frac{\left(x-x'\right)^2}{4Dt}} - e^{-\frac{\left(x+x'\right)^2}{4Dt}}}{\sqrt{4\pi Dt}}. \quad (24)$$

Note, that in this case the probability decays with time since particles that cross the boundary are absorbed there.

The probability that the particle is still in the positive side of the x axis is:

$$S\left(x_0, t\right) = \int_0^\infty \frac{e^{-\frac{\left(x-x_0\right)^2}{4Dt}} - e^{-\frac{\left(x+x_0\right)^2}{4Dt}}}{\sqrt{4\pi Dt}} dx = Erf\left(\frac{x_0}{\sqrt{4Dt}}\right). \quad (25)$$

The first passage time PDF is given by

$$f\left(x_0, t\right) = -\frac{dS}{dt} = \frac{x_0 e^{-\frac{-x_0^2}{4Dt}}}{t^{3/2}\sqrt{4\pi D}}. \quad (26)$$

One can easily see that the PDF has a power-law tail. Although the probability to cross the boundary is 1, as apparent from the fact that $\int\limits_{0}^{\infty} f\left(x_0, t\right) dt = 1$, the average crossing time diverges.