The Stable Levy Distributions

The Levy distributions are usually defined using their Fourier or Laplace transform. However, some of their properties can be derived without numerically inverting them.

The Fourier transform of the symmetric Levy distribution is given by $e^{-A|k|^{\alpha}}$. For convenience, in what follows we will set $A = 1$ and a generalization of the results for the case of $A \neq 1$ is straightforward.

(You may want to prove to yourself that $l_{\alpha,A,0}(x) = A^{-1/\alpha} l_{\alpha,1,0}\left(\frac{x}{A^{1/\alpha}}\right)$).

Inverting the Fourier transform we write $l_{\alpha,1,0}(x)$ as

$$l_{\alpha,1,0}(x) = \frac{1}{\pi}\int_0^{\infty} \cos(kx)\, e^{-|k|^{\alpha}}\, dk. \tag{1}$$

Integrating by parts we get,

$$\begin{aligned} l_{\alpha,1,0}(x) &= \frac{1}{\pi}\left[e^{-k^{\alpha}}\frac{1}{x}\sin(kx)\Big|_{k=0}^{k=\infty} + \frac{\alpha}{x}\int_0^{\infty}\sin(kx)\, k^{\alpha-1}e^{-k^{\alpha}}\, dk\right] &(2)\\ &= \frac{1}{\pi}\frac{\alpha}{x}\int_0^{\infty}\sin(kx)\, k^{\alpha-1}e^{-k^{\alpha}}\, dk \\ u &= e^{-k^{\alpha}} \rightarrow du = -\alpha k^{\alpha-1}e^{-k^{\alpha}}\, dk \\ dv &= \cos(kx)\, dk \rightarrow v = \frac{1}{x}\sin(kx) \end{aligned}$$

Changing variables according to $q = kx$ we get

$$l_{\alpha,1,0}(x) = \frac{1}{\pi}\frac{\alpha}{x^{1+\alpha}}\int_0^{\infty}\sin(q)\, q^{\alpha-1}e^{-\left(\frac{q}{x}\right)^{\alpha}}\, dq \tag{3}$$

For large x and $0 < \alpha < 1$

$$l_{\alpha,1,0}(x) \sim \frac{1}{\pi}\frac{\alpha}{x^{1+\alpha}}\int_0^{\infty}\sin(q)\, q^{\alpha-1}dq = \frac{1}{\pi}\frac{\alpha\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)}{x^{1+\alpha}}. \tag{4}$$

It is easy to see that the above asymptotic behavior is also valid for the Lorentzian case of $\alpha = 1$. For $1 < \alpha < 2$ the integral in the equation above diverges and we will use a different approach. We turn back to equation (1) and change variables according to $\omega = kx$ and $\eta = x^{-\alpha}$

$$l_{\alpha,1,0}(x) = \frac{\eta^{\frac{1}{\alpha}}}{\pi}\int_0^{\infty} e^{-\eta\omega}\cos(\omega)\, e^{-\eta(\omega^{\alpha}-\omega)}\, d\omega \tag{5}$$

For large x (hence, small η) we can expand the integrand

$$l_{\alpha,1,0}(x) \sim \frac{1}{\pi x}\int_0^{\infty} e^{-\eta\omega}\cos(\omega)\left[1 - \eta(\omega^{\alpha}-\omega) + ...\right] d\omega \sim \frac{1}{\pi}\frac{\alpha\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)}{x^{1+\alpha}} \tag{6}$$

It is also possible to derive the asymptotic behavior of the 1 sided Levy PDF. This PDF is defined using its Laplace transform which is of the form $e^{-u^{\alpha}}$ (where u is real and $0 < \alpha < 1$, again we set the coefficient $A = 1$). Inverting the Laplace transform we write,

$$l_{\alpha,1,1}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ut-u^{\alpha}}\, du. \tag{7}$$

Changing variables according to $z=-ut$ we write

$$l_{\alpha,1,1}(t)=\frac{1}{2\pi ti}\int_{\overline{c}-i\infty}^{\overline{c}+i\infty}e^{-z-\left(-\frac{z}{t}\right)^{\alpha}}dz. \tag{8}$$

For large t, i.e., $1/t^{\alpha}<<1$ we can expand the exponent in a Taylor series

$$l_{\alpha,1,1}(t)=\frac{1}{2\pi ti}\sum_{n=0}^{\infty}\frac{\left(-1/t^{\alpha}\right)^{n}}{n!}\int_{\overline{c}-i\infty}^{\overline{c}+i\infty}(-z)^{\alpha n}e^{-z}dz. \tag{9}$$

The integral in the equation above is the Hankel representation of the Gamma function ($\frac{1}{\Gamma(-\alpha)}=\frac{i}{2\pi}\int_{\overline{c}-i\infty}^{\overline{c}+i\infty}(-z)^{\alpha}e^{-z}dz$).

Thus we get,

$$l_{\alpha,1,1}(t)=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!t^{\alpha n+1}\Gamma(-\alpha n)} \tag{10}$$

Using the reflection formula for the Gamma function

$$\Gamma(z)\Gamma(1-z)=\pi\csc(\pi z) \tag{11}$$

we find

$$l_{\alpha,1,1}(t)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\Gamma(1+\alpha n)}{n!}\sin(n\pi\alpha)\,t^{-(1+n\alpha)}.$$

The $n=0$ term does not contribute for a finite t (it can be seen by the fact the inverse Laplace transform of 1 is a delta function or by the fact that $\Gamma(0)$ diverges). The long time limit is given by

$$l_{\alpha,1,1}(t)\sim\frac{1}{t^{\alpha+1}\left|\Gamma(-\alpha)\right|}=\frac{\Gamma(1+\alpha)\sin(\pi\alpha)}{\pi}t^{-(1+\alpha)}. \tag{12}$$

The small t behavior may be found using the steepest descent method leading to:

$$l_{\alpha,1,1}(t)\sim Bt^{-\sigma}e^{-\kappa t^{-\tau}}$$

where,

$$\begin{aligned}\sigma &= \frac{2-\alpha}{2(1-\alpha)}\\ \kappa &= (1-\alpha)\,\alpha^{\alpha/(1-\alpha)}\\ \tau &= \frac{\alpha}{1-\alpha}\end{aligned} \tag{13}$$

Random walks

Example 1:

Consider a 1D random walk with iid steps characterized by the following PDF:

$$p(x)=\frac{A}{a+\cosh(x)}. \tag{14}$$

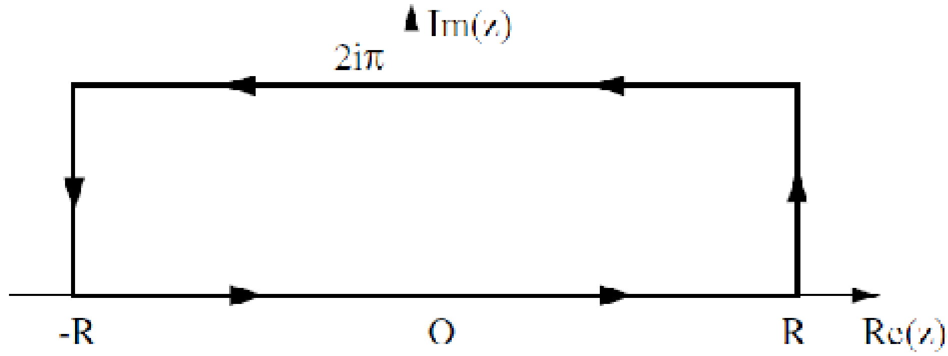

Figure 1: Integration contour for the calculation of the characteristic function (eq. 15)

This PDF is exponentially wider than a Gaussian (for large x it decays like $p\left(x\right)\sim Ae^{-x}$ compared with the $p\left(x\right)\sim Be^{-x^2}$ of the Gaussian distribution) but still has all the moments finite. For simplicity we consider the case of $|a|<1$. The normalization factor, A, will be determined later. The characteristic function is:

$$\widetilde{p}\left(k\right)=\int\limits_{-\infty}^{\infty}e^{-ikx}p(x)dx=\int\limits_{-\infty}^{\infty}e^{-ikx}\frac{A}{a+\cosh\left(x\right)}dx. \tag{15}$$

The integral can be solved by integration in the complex plain along the contour shown in figure 1.

The integral along the lower line is $\widetilde{p}\left(k\right)$, the integral along the upper line is $-e^{2\pi k}\widetilde{p}\left(k\right)$ (the *cosh* is periodic in the complex plain with a period of $2\pi i$), the integral along the vertical lines vanishes in the limit $R\rightarrow\infty$ (the integral along the vertical lines is bounded by

$$\left|\int\frac{e^{-ikz}}{a+\cosh\left(z\right)}dz\right|\leq\int\left|\frac{e^{-ikz}}{a+\cosh\left(z\right)}\right|dz\leq2\pi i\frac{e^{-ikR}e^{2\pi k}}{\frac{e^{R}+e^{-R}}{2}-a}, \tag{16}$$

which vanishes for $R=\pm\infty$).

The final expression is:

$$\oint\frac{e^{-ikz}}{a+\cosh\left(z\right)}dz=\frac{1}{A}\widetilde{p}\left(k\right)\left(1-e^{2\pi k}\right)=2\pi i\sum_{m}res\left(z_m\right), \tag{17}$$

where z_m are the poles within the contour. The poles are given by

$$a+\cosh\left(z\right)=0,$$

or defining $e^{z}=w$

$$\begin{aligned} 0 &= 2aw+w^2+1,\\ w_{\pm} &= -a\pm\sqrt{a^2-1},\\ w_{\pm} &= e^{i(\pi\pm\beta)} \text{ where } \beta=\cos^{-1}a. \end{aligned}$$

One can easily see that $z_{\pm}=i\left(\pi\pm\beta\right)$ are indeed the poles,

$$\cosh\left(z_{\pm}\right)=\frac{e^{i(\pi\pm\beta)}+e^{-i(\pi\pm\beta)}}{2}=-\frac{1}{2}\left(e^{i(\pm\beta)}+e^{-i(\pm\beta)}\right)=-\cos\beta=-\cos\left(\cos^{-1}a\right)=-a.$$

The residuals are:

$$res(z_{\pm}) = e^{-ikz_{\pm}} \lim_{z \to z_{\pm}} \frac{z - z_{\pm}}{\cosh(z) + a} = e^{-ikz_{\pm}} \lim_{z \to z_{\pm}} \frac{1}{\frac{d}{dz}(\cosh(z) + a)} = e^{-ikz_{\pm}} \lim_{z \to z_{\pm}} \frac{1}{\sinh(z)}$$

$$\begin{aligned} res(z_{\pm}) &= \frac{e^{-ikz_{\pm}}}{\sinh(z_{\pm})} = \frac{e^{-iki(\pi \pm \beta)}}{\sinh(i(\pi \pm \beta))} = \frac{e^{k(\pi \pm \beta)}}{i \sin(\pi \pm \beta)} = \frac{e^{k(\pi \pm \beta)}}{i(\sin(\pi)\cos(\beta) + \cos(\pi)\sin(\pm\beta))} \\ &= \pm \frac{e^{k(\pi \pm \beta)}}{-i \sin(\beta)}. \end{aligned}$$

Using the relation between the contour integral and the characteristic function (Eq. 17) we can write,

$$\frac{1}{A} \widetilde{p}(k) \left(1 - e^{2\pi k}\right) = -2\pi e^{k\pi} \frac{e^{k\beta} - e^{-k\beta}}{\sin(\beta)} = -2\pi e^{k\pi} \frac{2 \sinh(k\beta)}{\sin(\beta)},$$

which yields,

$$\widetilde{p}(k) = 2\pi \frac{\sinh(k\beta)}{\sin(\beta)} \frac{A}{\sinh(k\pi)}.$$

The normalization requires that $\widetilde{p}(0) = 1$,

$$\widetilde{p}(k = 0) = \frac{2\beta}{\sin(\beta)} A = 1 \to A = \frac{\sin(\beta)}{2\beta}.$$

The characteristic function is given by:

$$\widetilde{p}(k) = \frac{\pi}{\beta} \frac{\sinh(k\beta)}{\sinh(k\pi)}. \tag{18}$$

This function has a Taylor expansion as:

$$\widetilde{p}(k) = 1 - \frac{k^2}{2!} m_2 + \frac{k^4}{4!} m_4 - \frac{k^6}{6!} m_6 + O\left(k^8\right),$$

where,

$$\begin{aligned} m_2 &= \frac{\left(\pi^2 - \beta^2\right)}{3}; \\ m_4 &= m_2 \frac{\left(7\pi^2 - 3\beta^2\right)}{5}; \\ m_6 &= \frac{m_2}{3} \left(-18\pi^2\beta^2 + 3\beta^4 + 31\pi^4\right). \end{aligned}$$

The cumulants generating function is given by:

$$C(k) = \ln(\widetilde{p}(k)) = -\frac{k^2}{2!} c_2 + \frac{k^4}{4!} c_4 - \frac{k^6}{6!} c_6 + O\left(k^8\right),$$

where,

$$\begin{aligned} c_2 &= m_2 = \frac{\left(\pi^2 - \beta^2\right)}{3}; \\ c_4 &= \frac{2}{5} m_2 \left(\beta^2 + \pi^2\right); \\ c_6 &= \frac{16}{21} m_2 \left(\pi\beta + \beta^2 + \pi^2\right) \left(-\pi\beta + \beta^2 + \pi^2\right); \end{aligned}$$

Following our derivation of the corrections to the Gaussian approximation, we may write the PDF for the sum of N iid steps taken from the above distribution (Eq. 14) to be:

$$p_z(Z) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2} \left(1 + \frac{\kappa}{4!N}\left[3 - 12\xi^2 + 4\xi^4\right] + O\left(\frac{1}{N^2}\right)\right), \tag{19}$$

where we defined

$$\begin{aligned} \xi &= Z/\sqrt{2}; Z = \frac{1}{\sigma\sqrt{N}} \sum_{i=1}^{N} x_i \\ \kappa &= \frac{c_4}{\sigma^4} = \frac{6}{5}\frac{\left(\pi^2 + \beta^2\right)}{\left(\pi^2 - \beta^2\right)}. \end{aligned} \tag{20}$$

Using the results above we find that

$$Z_{edge} = (4!)^{1/4} \left(\frac{5}{6}\frac{\left(\pi^2 - \beta^2\right)}{\left(\pi^2 + \beta^2\right)}\right)^{1/4} N^{1/4}.$$

Note that the positive kurtosis implies that the distribution is wider than a Gaussian PDF, as expected from the fact that the PDF of a single step is wider than a Gaussian PDF.

Example 2:

Consider a 1D random walk with iid steps characterized by the following PDF:

$$p(x) = \frac{A}{1 + x^4}. \tag{21}$$

This PDF has very fat tails, and all the moments higher than 2 diverge (including $\left\langle |x|^3 \right\rangle$, which is used in the Berry-Esseen theorem). The characteristic function is,

$$\widetilde{p}(k) = A \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1 + x^4} dx. \tag{22}$$

We can calculate the integral by integration in the complex plain along the contour shown in Figure 2 (for $k < 0$ and by the symmetry we can find the function for $k > 0$).

The integral on the radial contour is zero according to:

$$\begin{aligned} \left| \int f(z)\, dz \right| &\leq \int |f(z)|\, dz, \\ \left| \int \frac{e^{-ikz}}{1 + z^4} dz \right| &\leq \pi R \frac{1}{R^4 - 1}; \\ \lim_{R\to\infty} \pi R \frac{1}{R^4 - 1} &= 0. \end{aligned}$$

Therefore, the characteristic function is given by

$$\widetilde{p}(k) = 2\pi i \left(res(z_1) + res(z_2)\right),$$

where $z_{1,2}$ are the roots of $1 + z^4 = 0$, with a positive imaginary part. That is,

$$z_1 = e^{i\frac{3\pi}{4}} = \frac{-1 + i}{\sqrt{2}}; z_2 = e^{i\frac{\pi}{4}} = \frac{1 + i}{\sqrt{2}}.$$

Therefore,

$$\widetilde{p}(k) = 2\pi i A \left(\frac{e^{-ik(1+i)/\sqrt{2}}}{4e^{i\frac{3\pi}{4}}} + \frac{e^{-ik(-1+i)/\sqrt{2}}}{4e^{i\frac{9\pi}{4}}}\right) = Ae^{k/\sqrt{2}} \frac{\pi}{\sqrt{2}} \left(\cos\left(\frac{k}{\sqrt{2}}\right) - \sin\left(\frac{k}{\sqrt{2}}\right)\right).$$

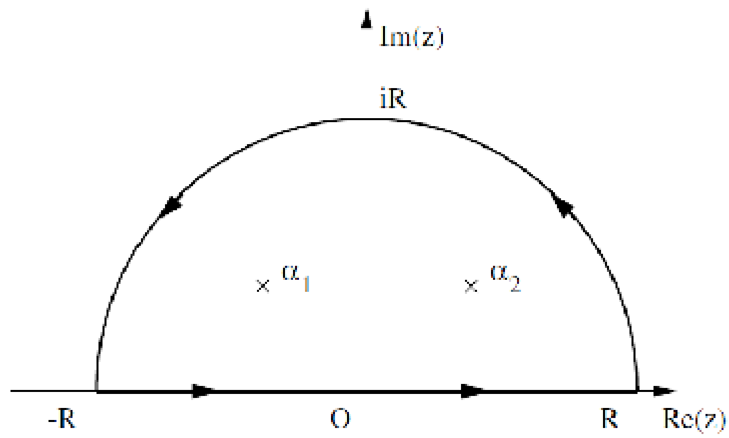

Figure 2: Integration contour for the calculation of the characteristic function (eq. 22)

The normalization requires that $\widetilde{p}(0) = 1$,

$$\widetilde{p}(k=0) = A\frac{\pi}{\sqrt{2}} = 1 \rightarrow A = \frac{\sqrt{2}}{\pi}.$$

The characteristic function for $k < 0$ is:

$$\widetilde{p}(k) = e^{k/\sqrt{2}}\left(\cos\left(\frac{k}{\sqrt{2}}\right) - \sin\left(\frac{k}{\sqrt{2}}\right)\right).$$

Using the symmetry of the function (writing $k = -|k|$ in the expression above) we find for any k

$$\widetilde{p}(k) = e^{-|k|/\sqrt{2}}\left(\cos\left(\frac{|k|}{\sqrt{2}}\right) + \sin\left(\frac{|k|}{\sqrt{2}}\right)\right). \tag{23}$$

Note that this function does not have a Taylor expansion since its moments $\langle |x|^{\mu}\rangle$, $\mu > 2$, diverge. However, there is the following asymptotic behavior:

$$\widetilde{p}(k) = \left(1 - \frac{|k|}{\sqrt{2}} + \frac{k^2}{4} - \frac{\sqrt{2}}{4\cdot 3!}|k|^3 + \frac{k^4}{4\cdot 4!} + O\left(k^5\right)...\right)\left(1 - \frac{k^2}{4} + \frac{k^4}{4\cdot 4!} ... + \frac{|k|}{\sqrt{2}} - \frac{\sqrt{2}\,|k|^3}{4\cdot 3!} + ... + O\left(k^5\right)\right)$$

$$\widetilde{p}(k) \sim 1 - \frac{k^2}{2} + \frac{|k|^3}{3\sqrt{2}} - \frac{k^4}{24} + O\left(k^5\right).$$

The first two terms correspond to the Gaussian behavior, and the third and higher order (in k) terms signal that the higher moments diverge (the absolute value of k rather than k). Similarly, the asymptotic behavior of the cumulants generating function is given by:

$$C(k) = \ln\left(\widetilde{p}(k)\right) \sim -\frac{k^2}{2} + \frac{|k|^3}{3\sqrt{2}} - \frac{k^4}{6} + O\left(k^5\right). \tag{24}$$

Note that the coefficient of k^4 is not the kurtosis because the fourth moment diverges.

The PDF for the position after N steps is given by

$$P_N(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\widetilde{p}(k)\right)^N e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{N\frac{|k|^3}{3\sqrt{2}} - ...}\right)e^{-k^2N/2}e^{ikx}dk.$$

Defining $w = k\sqrt{N}$, the integral becomes:

$$P_N(x) = \frac{1}{2\pi\sqrt{N}} \int_{-\infty}^{\infty} \left(e^{\frac{|w|^3}{3\sqrt{2N}} - O\left(\frac{1}{N}\right)\cdots} \right) e^{-w^2/2} e^{iwx/\sqrt{N}} dw,$$

or defining $\varsigma = x/\sqrt{N}$ (and therefore,$P_N(x)\frac{dx}{d\varsigma} = P_N(x)\sqrt{N} = P_N(\varsigma)$)

$$P_N(\varsigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{\frac{|w|^3}{3\sqrt{2N}} - O\left(\frac{1}{N}\right)\cdots} \right) e^{-w^2/2} e^{iw\varsigma} dw,$$

which suggests that the CLT holds.

The leading order correction to the Gaussian, in the limit of large N, is calculated by expanding the expression in the parenthesis

$$\begin{aligned} P_N(\varsigma) &\sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 + \frac{|w|^3}{3\sqrt{2N}} \right) e^{-w^2/2} e^{iw\varsigma} dw \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-w^2/2} e^{iw\varsigma} dw + \frac{1}{2\pi} \frac{1}{3\sqrt{2N}} \left((-i)^3 \frac{\partial^3}{\partial\varsigma^3} \right) \int_{-\infty}^{\infty} sgn(w)\, e^{-w^2/2} e^{iw\varsigma} dw \\ &= \frac{1}{\sqrt{2\pi}} e^{-\varsigma^2/2} + \frac{1}{2\pi} \frac{1}{3\sqrt{2N}} \left(i \frac{\partial^3}{\partial\varsigma^3} \right) \int_{-\infty}^{\infty} sgn(w)\, e^{-w^2/2} e^{iw\varsigma} dw. \end{aligned}$$

The integral may be written as:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} sgn(w)\, e^{-w^2/2} e^{iw\varsigma} dw = \frac{1}{2\pi} 2i \int_0^{\infty} \sin(w\varsigma)\, e^{-w^2/2} dw = \frac{-i}{\pi} \operatorname{Im} \left[\int_0^{\infty} e^{-w^2/2} e^{-iw\varsigma} dw \right] \tag{25}$$

We define,

$$I \equiv \int_0^{\infty} e^{-w^2/2} e^{-iw\varsigma} dw = e^{-\varsigma^2/2} \int_0^{\infty} e^{-(w+i\varsigma)^2/2} dw = e^{-\varsigma^2/2} \int_{i\varsigma}^{\infty + i\varsigma} e^{-z^2/2} dw \tag{26}$$

We use the contour shown in Figure 3 to calculate the integral of $e^{-z^2/2}$.

The right segment goes to zero as $R \to \infty$ (it's bounded by $\varsigma e^{-R^2/2}$). The upper segment is $e^{\varsigma^2/2} I$ the lower segment is a real integral, and the left segment is the special feature of this problem due to the $sgn(w)$. The integral along the contour is zero because there are no poles. Therefore,

$$\begin{aligned} 0 &= \oint e^{-z^2/2} dz = e^{\varsigma^2/2} I - \int_0^{\infty} e^{-w^2/2} dw + i \int_0^{\varsigma} e^{w^2/2} dw; \\ I &= e^{-\varsigma^2/2} \left(\int_0^{\infty} e^{-w^2/2} dw - i \int_0^{\varsigma} e^{w^2/2} dw \right) = e^{-\varsigma^2/2} \left(\frac{\sqrt{\pi}}{2} - i \int_0^{\varsigma} e^{w^2/2} dw \right). \end{aligned}$$

Substituting it back into the original integral (Eq. 25) we obtain,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} sgn(w)\, e^{-w^2/2} e^{iw\varsigma} dw &= \frac{-i}{\pi} \operatorname{Im} \left[\int_0^{\infty} e^{-w^2/2} e^{-iw\varsigma} dw \right] = \frac{-i}{\pi} \operatorname{Im} \left[e^{-\varsigma^2/2} \left(\frac{\sqrt{\pi}}{2} - i \int_0^{\varsigma} e^{w^2/2} dw \right) \right] \\ &= \frac{i}{\pi} e^{-\varsigma^2/2} \int_0^{\varsigma} e^{w^2/2} dw, \end{aligned}$$

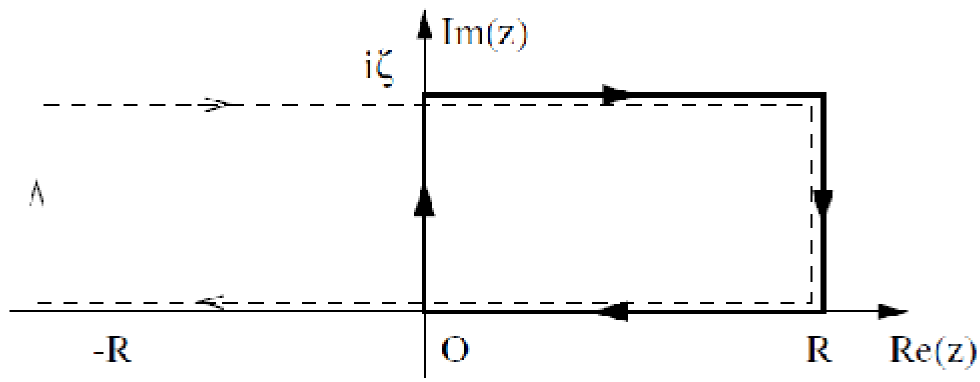

Figure 3: Contour for the integration of $e^{-z^2/2}$, in order to calculate I (eq. 26)

the latter integral may be expressed in terms of the Dawson's integral

$$\begin{aligned} D\left(x\right) &= e^{-x^2}\int\limits_0^x e^{y^2}dy \sim \frac{1}{2x}\left(1+\frac{1}{2x^2}+\frac{3}{4x^4}+\sum_{m=3}^{\infty}\frac{(2m-1)!!}{2^m x^{2m}}\right); \\ m!! &\equiv \begin{array}{ll} \prod\limits_{l=1}^{m/2}(2l) & l \text{ is even} \\ \prod\limits_{l=1}^{(m+1)/2}(2l-1) & l \text{ is odd} \end{array}. \end{aligned}$$

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} sgn\left(w\right)e^{-w^2/2}e^{iw\varsigma}dw = \frac{i\sqrt{2}}{\pi}D\left(\frac{\varsigma}{\sqrt{2}}\right). \qquad (27)$$

Note that from the definition of $D\left(x\right)$ one may obtain:

$$\begin{aligned} D'\left(x\right) &= -2xD\left(x\right)+1; \\ D''(x) &= -2xD'\left(x\right)-2D\left(x\right) = -2\left(\left(1-2x^2\right)D\left(x\right)+x\right) \\ D'''\left(x\right) &= -2(1-4xD\left(x\right)+\left(1-2x^2\right)D'\left(x\right)) = -4(1-x^2-x\left(3-2x^2\right)D\left(x\right)) \end{aligned}$$

Substituting these relations in the expression for the PDF one obatins:

$$\begin{aligned} P_N\left(\varsigma\right) &\sim \frac{1}{\sqrt{2\pi}}e^{-\varsigma^2/2}-\frac{1}{3\pi\sqrt{N}}\left(\frac{\partial^3}{\partial\varsigma^3}\right)D\left(\frac{\varsigma}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\varsigma^2/2}+\frac{2}{3\pi\sqrt{2N}}\left(1-\frac{\varsigma^2}{2}+\left(\frac{\varsigma^3}{\sqrt{2}}-3\frac{\varsigma}{\sqrt{2}}\right)D\left(\frac{\varsigma}{\sqrt{2}}\right)\right), \end{aligned}$$

where we used

$$\frac{\partial}{\partial\varsigma} = \frac{\partial}{\partial\frac{\varsigma}{\sqrt{2}}}\frac{\partial\frac{\varsigma}{\sqrt{2}}}{\partial\varsigma} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\frac{\varsigma}{\sqrt{2}}}.$$

For large ς we have

$$D\left(\frac{\varsigma}{\sqrt{2}}\right) \sim \frac{1}{2\frac{\varsigma}{\sqrt{2}}}\left(1+\frac{1}{2\left(\frac{\varsigma}{\sqrt{2}}\right)^2}+\frac{3}{4\left(\frac{\varsigma}{\sqrt{2}}\right)^4}+\sum_{m=3}^{\infty}\frac{(2m-1)!!}{2^m\left(\frac{\varsigma}{\sqrt{2}}\right)^{2m}}\right).$$

Therefore,

$$P_N(\varsigma) \sim \frac{1}{\sqrt{2\pi}} e^{-\varsigma^2/2} - \frac{2}{\pi\sqrt{2N}} \frac{1}{\varsigma^4} + O\left(\varsigma^{-6}\right).$$

The Gaussian approximation breaks when the correction term is of the same order of the Gaussian term, i.e.,

$$\begin{aligned} e^{-\varsigma_{edge}^2/2} &\approx \frac{2}{\sqrt{\pi N}} \left(\frac{1}{\varsigma_{edge}^4}\right) \\ -\varsigma_{edge}^2/2 &= -2\ln\varsigma_{edge}^2 - \frac{1}{2}\ln N + .. \\ \varsigma_{edge}^2 &= 4\ln\varsigma_{edge}^2 + \ln N. \end{aligned}$$

Recursively substituting the LHS in the RHS, one obtains,

$$\begin{aligned} \varsigma_{edge}^2 &= \ln N + 4\ln\ln N + ... \\ \varsigma_{edge} &= \sqrt{\ln N}\left(1 + 2\frac{\ln\ln N}{\ln N} + ...\right). \end{aligned}$$

Note that ς_{edge} is very small even for a very large N. This slow convergence is due to the wide PDF of the steps. Therefore, care must be taken when using the CLT with fat tailed PDFs. Note also that the leading order correction term to the Gaussian PDF is proportional to $\frac{1}{\varsigma^4}$ due to the original PDF which decays like $\frac{1}{x^4}$.