כבידה 1

2021/22 סמסטר א'

# <u>פתרון תרגיל 1</u>

## **1. הרכבה של טרנספורמציות לורנץ**

The product of the transformations is

$$\begin{bmatrix} \gamma_x & \gamma_x\beta_x & 0 & 0 \\ \gamma_x\beta_x & \gamma_x & 0 & 0 \\ & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_y & 0 & \gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_x\beta_x & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_x\gamma_y & \gamma_x\beta_x & \gamma_x\gamma_y\beta_y & 0 \\ \gamma_x\gamma_y\beta_x & \gamma_x & \gamma_x\gamma_y\beta_x\beta_y & 0 \\ \gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

but

$$\begin{bmatrix} \gamma_y & 0 & \gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_x\beta_x & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_x & \gamma_x\beta_x & 0 & 0 \\ \gamma_x\beta_x & \gamma_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_x\gamma_y & \gamma_y\gamma_x\beta_x & \gamma_y\beta_y & 0 \\ \gamma_x\beta_x & \gamma_x & 0 & 0 \\ \gamma_x\gamma_y\beta_y & \gamma_x\gamma_y\beta_x\beta_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is not the same.

It should remind you of rotations in 3 dimensional space.

## 2. טרנספורמצית לורנץ בחלל

(a) According to the formula for a Lorentz transformation

$$\Delta t_{observer} = \gamma\left(\Delta t_{Earth-Sun} - \frac{u}{c^2}\Delta x_{Earth-Sun}\right), \gamma = \frac{1}{\sqrt{1-(\frac{u}{c})^2}}.$$

Plugging in the numbers gives

$$\Delta t_{observer} = \frac{2min - 0.8(8.3min)}{\sqrt{1-0.8^2}} = -7.7min.$$

This means that according to the observer, event B happened before event A! If we reverse the sign of u then

$$\Delta t_{observer} = \frac{2min + 0.8(8.3min)}{\sqrt{1-0.8^2}} = 14min.$$

(b) According to an observer on the spacecraft, $\Delta x_{observer} = 0$.So we can write

$$\begin{aligned} \Delta x_{Earth-Sun} &= \gamma(0 + v\Delta t_{observer}) \\ \frac{\Delta x_{Earth-Sun}}{\Delta t_{observer}} &= v\gamma = \frac{v}{\sqrt{1-(\frac{v}{c})^2}} = \frac{c}{\sqrt{(\frac{c}{v})^2 - 1}} \end{aligned}$$

and solving for v gives

$$\begin{aligned} v &= c\left[1 + \left(\frac{\Delta t_{observer}}{\Delta x_{Earth-Sun/c}}\right)^2\right]^{-1/2} \\ &= (3\times 10^8 m/s)\left[1+\left(\frac{5min}{8.3min}\right)^2\right]^{-1/2} = 2.6\times 10^8 m/s. \end{aligned}$$

(c) In the Earth-Sun frame

$$\Delta t_{Earth-Sun} = \frac{\Delta x_{Earth-Sun}}{v} = \frac{8.3light - minutes}{2.6\times 10^8 m/s} = 9.6min.$$

Alternately we can use the time dilation formula to get (the difference is due to rounding errors)

$$\Delta t_{Earth-Sun} = \frac{\Delta x_{Earth-Sun}}{\sqrt{1-(v/c)^2}} = \frac{5min}{\sqrt{1-\left[1+(\frac{5min}{8.3min})^2\right]^{-1}}} = 9.7min$$

## 3. שינויי זמן ומרחק

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \simeq 1 + \frac{1}{2}\beta^2$$

where $\beta = v/c$ and we approximate for small $\beta$ because the plane is moving much more slowly than light.

Take $\Delta t$ as the time interval measured by an observer at rest in S. Then

$$\Delta t = \gamma \Delta t'$$

(just plug in a Lorentz transformation with $x = 0$ for rest frame). Then

$$\delta t = \Delta t - \Delta t' \simeq \frac{\beta^2}{2} \Delta t'$$

and for $v = 400\frac{m}{s}$ and $\Delta t' = 3600s$ the difference $\delta t = 3.2ns$.

(b) We assume the meter stick is at rest in S'. As observed by stationary observers in S, the stick moves in the positive x direction with speed v. $x' = \gamma(x - vt)$ relates the position x' measured in S' with the position x measured in X.

Let $\Delta x'$ be the length of the stick measured by an observer at rest in S' (the "proper length"). The stick is moving with speed v in S. To find its length in S, the positions of the front and back of the stick are observed at the same time by two stationary observers in S. The length they measure is the distance $\Delta x$ between them at $\Delta t = 0$.Then

$$\Delta x' = \gamma \Delta x$$

$$\beta = \sqrt{1 - \left(\frac{\Delta x}{\Delta x'}\right)^2}$$

and for $\Delta x = \Delta x'/2$ you get $\beta = 0.866$. The length of the stick as measured by observers at rest in S is smaller than the length measured by an observer at rest with respect to the stick. ("Lorentz contraction..")

## 4. בריחה של צופה מכוכב

Solving the EOM for an observer at infinity :

$$\dot{r} = \sqrt{\frac{2GM}{r}} \Rightarrow t = \frac{1}{\sqrt{2GM}} \int_R^r dr \sqrt{r} = \frac{1}{3}\sqrt{\frac{2}{GM}}\left(r^{3/2} - R^{3/2}\right)$$

In observer A frame we should put notice to its velocity (and the velocity change)

$$dt' = dt\sqrt{1-v^2} \Rightarrow t' = \frac{1}{\sqrt{2GM}} \int_R^r dr \sqrt{r\left(1 - \frac{2GM}{r}\right)}$$

$$= \frac{1}{\sqrt{2GM}} \int_R^r dr \sqrt{r - 2GM}$$

$$= \frac{1}{3}\sqrt{\frac{2}{GM}}\left((r-2GM)^{3/2} - (R-2GM)^{3/2}\right)$$

$$\approx \frac{1}{3}\sqrt{\frac{2}{GM}}\left(r^{3/2} - 3GMr^{1/2} - R^{3/2} + 3GMR^{1/2}\right)$$

$$\Delta t = t - t' \approx \frac{1}{3}\sqrt{\frac{2}{GM}}\left(3GMr^{1/2} - 3GMR^{1/2}\right)$$

$$= \sqrt{2GM}\left(\sqrt{r} - \sqrt{R}\right) \approx \sqrt{2GMr}$$

B. putting c back, we get:

$$\Delta t \approx \frac{1}{c^2}\sqrt{2GMr} \approx 100\mu\sec$$

## 5. חלליות החלל

Take one rocket (call it rocket 1) and consider how fast the other rocket (rocket 2) looks in this frame. In the Earth frame, rocket 1 has velocity c/2 and rocket 2 has velocity -c/2. Applying the velocity addition law gives

$$v_2' = \frac{v_2 - v_1}{1 - v_1 v_2/c^2} = \frac{(-c/2) - (c/2)}{1 - (c/2)(-c/2)/c^2} = -\frac{4}{5}c$$

so rocket 2 looks like it is approaching at 4/5 c. Applying the Lorentz contraction formula gives

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}L_0.$$

## 6. חיבור מהירויות

Particle velocity is $u_z$, so its spacetime position in its own system $S$ is

$$x^\mu = (ct, 0, 0, ut)$$

and its position in S' is

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

$$\begin{pmatrix} ct' \\ x' \\ y \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Thus

$$z' = \gamma t(v + u)$$

but this is a function of t not of t'. In the S' system, plugging in $\gamma t$

$$z' = \frac{(v+u)}{1 + \frac{vu}{c^2}}t'$$

and differentiating by t'

$$\frac{dz'}{dt'} = u_z' = \frac{(v+u)}{1 + \frac{vu}{c^2}}$$

which is the same formula you saw in class.

## 7. שדה אלקטרומגנטי

# 1 Lorentz Transformation of the Electromagnetic Field

The electromagnetic field tensor $F^{\mu\nu}$, also called the *field strength* tensor of the electromagnetic field, in an inertial frame is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{1}$$

It is an *antisymmetric* tensor

$$F^{\mu\nu} = -F^{\nu\mu} \tag{2}$$

What is the magnetic field $B^{i'}$ in another inertial frame, moving with relative velocity $v$ along the $x$ axis?

To find out, we transform the tensor $F^{\mu\nu}$ and look at the desired components. A $\binom{2}{0}$-tensor transformation rule is

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}F^{\rho\sigma} \tag{3}$$

where

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

We work with units $c = 1$, so $\beta = v$. When making index calculation we notice which components vanish, so not to write the whole sum explicitly.

The component parallel to $v$ transforms

$$\begin{aligned} B'_x = F^{2'3'} &= \Lambda^{2'}{}_{\rho}\Lambda^{3'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{2'}{}_{2}\Lambda^{3'}{}_{3}F^{23} = B_x \end{aligned} \tag{5}$$

The $y$ - component transforms

$$\begin{aligned} B'_y = F^{3'1'} &= \Lambda^{3'}{}_{\rho}\Lambda^{1'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{3'}{}_{3}\Lambda^{1'}{}_{0}F^{30} + \Lambda^{3'}{}_{3}\Lambda^{1'}{}_{1}F^{31} \\ &= -\gamma v\left(-E_z\right) + \gamma B_y = \gamma\left(B_y - \left(v_z E_x - v_x E_z\right)\right) \\ &= \gamma\left(\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}\right)_y \end{aligned} \tag{6}$$

The $z$ - component transforms

$$\begin{aligned} B'_z = F^{1'2'} &= \Lambda^{1'}{}_{\rho}\Lambda^{2'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{1'}{}_{0}\Lambda^{2'}{}_{2}F^{02} + \Lambda^{1'}{}_{1}\Lambda^{2'}{}_{2}F^{12} \\ &= -\gamma v E_y + \gamma B_z = \gamma\left(B_z - \left(v_x E_y - v_y E_x\right)\right) \\ &= \gamma\left(\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}\right)_z \end{aligned} \tag{7}$$

where we used the fact that

$$v^i = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

and

$$(\boldsymbol{v} \times \boldsymbol{E})^i = \begin{pmatrix} v_y E_z - v_z E_y \\ v_z E_x - v_x E_y \\ v_x E_y - v_y E_x \end{pmatrix} \tag{9}$$

We conclude, in general that

$$\boldsymbol{B}'_{\shortparallel} = \boldsymbol{B}_{\shortparallel} \tag{10}$$

$$\boldsymbol{B}'_{\perp} = \gamma \left( \boldsymbol{B}_{\perp} - \boldsymbol{v} \times \boldsymbol{E} \right) \tag{11}$$

Lorentz transformation of the magnetic field

where $\shortparallel$ and $\perp$ mean parallel and perpendicular to $\boldsymbol{v}$. These are 3-vector equations, so they have the same form when rotating the system.

# 2 Lagrangian Density

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} \tag{12}$$

$$F_{0i} = \eta_{0\rho}\eta_{i\sigma}F^{\rho\sigma} = -F^{0i} = -E_i \tag{13}$$

$$F_{ij} = \eta_{i\rho}\eta_{j\sigma}F^{\rho\sigma} = F^{ij} \tag{14}$$

Open the summation, with $i \neq j$,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \\
&= -\frac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} + F_{ji}F^{ji}\right) \\
&= -\frac{1}{4}\left(F_{0i}F^{0i} + (-F_{0i})\left(-F^{0i}\right) + F_{ij}F^{ij} + (-F_{ij})\left(-F^{ij}\right)\right) \\
&= -\frac{1}{4}\left(2F_{0i}F^{0i} + 2F_{ij}F^{ij}\right) \\
&= -\frac{1}{4}\left(2\left(-E_i\right)E^i + 2\left((F_{12})^2 + (F_{13})^2 + (F_{23})^2\right)\right) \\
&= -\frac{1}{4}\left(-2\boldsymbol{E}^2 + 2\boldsymbol{B}^2\right) = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)
\end{aligned} \tag{15}$$

Lorentz transformations of the tensors

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}F^{\rho\sigma} \tag{16}$$

$$F_{\mu'\nu'} = \Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}F_{\alpha\beta} \tag{17}$$

where $\Lambda^{\alpha}{}_{\mu'}$ is the inverse matrix of $\Lambda^{\mu'}{}_{\rho}$,

$$\Lambda^{\alpha}{}_{\mu'}\Lambda^{\mu'}{}_{\rho} = \delta^{\alpha}_{\rho} \tag{18}$$

Therefore the full contraction of the tensors is invariant under Lorentz transformation (it is a “Lorentz scalar”)

$$
\begin{aligned}
F_{\mu'\nu'}F^{\mu'\nu'} &= \Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}\Lambda^{\mu'}{}_{\rho}\Lambda^{\upsilon'}{}_{\sigma}F_{\alpha\beta}F^{\rho\sigma} \\
&=\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}F_{\alpha\beta}F^{\rho\sigma} = F_{\rho\sigma}F^{\rho\sigma} = F_{\mu\nu}F^{\mu\nu}
\end{aligned}
\tag{19}
$$