Gravity 1 - Recitation 6

November 22, 2021

Contents

1 The Hyperbolic Plane

The *hyperbolic plane* can be defined by the metric

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right) \tag{1}$$

$$y > 0 \tag{2}$$

It is a classic example of non-Euclidean two-dimensional surface (this is a particular **model** of the hyperbolic plane, called the *Poincare half plane.*).
1. Show that points on the x-axis are an infinite distance from any other point (x, y) in the upper half-plane.
2. Calculate the Christoffel symbols directly from the metric, and write down the geodesic equations.
3. Derive the geodesic equations by a variational principle from the Lagrangian, and read off the Christoffel symbols.
4. Use integrals of motion to find the algebraic form of the geodesics, and see that they are semi-circles centered on the x-axis or vertical lines.
5. Find x and y as functions of the length parameter s along the geodesics.

1.1 Infinite Distance

The distance along a curve C of constant x (vertical line with $dx = 0$), towards a point on the x-axis ($y = 0$) is

$$\int_C ds = \int_{y_p}^{0} \frac{dy}{y} = ln\,(y)\,|_{y_p}^{0} \tag{3}$$

which diverges. Similarly, the distance along any other curve to a point on the x-axis will diverge.

1.2 Direct Calculation of the Christoffel Symbols

From (1) we read off he metric g_{ij}

$$\begin{aligned} g_{xx} = g_{yy} &= \frac{1}{y^2} \\ g_{xy} = g_{yx} &= 0 \end{aligned} \tag{4}$$

The inverse metric g^{ij} is

$$
\begin{aligned}
g^{xx} &= g^{yy} = y^2 \\
g^{xy} &= g^{yx} = 0
\end{aligned}
\tag{5}
$$

The formula for Christoffel symbols is

$$
\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \tag{6}
$$

Since the metric is diagonal,

$$
\Gamma^x_{ij} = \frac{1}{2} g^{xx} \left(\partial_i g_{xj} + \partial_j g_{xi} - \partial_x g_{ij} \right) \tag{7}
$$

$$
\Gamma^y_{ij} = \frac{1}{2} g^{yy} \left(\partial_i g_{yj} + \partial_j g_{yi} - \partial_y g_{ij} \right) \tag{8}
$$

The metric also does not depend on x, so the different components of (7) are

$$
\begin{aligned}
\Gamma^x_{xx} &= \frac{1}{2} g^{xx} \left(\partial_x g_{xx} + \partial_x g_{xx} - \partial_x g_{xx} \right) = 0 \\
\Gamma^x_{yy} &= \frac{1}{2} g^{xx} \left(\partial_y g_{xy} + \partial_y g_{xy} - \partial_x g_{yy} \right) = 0 \\
\Gamma^x_{xy} &= \frac{1}{2} g^{xx} \left(\partial_x g_{xy} + \partial_y g_{xx} - \partial_x g_{xy} \right) = \frac{1}{2} g^{xx} \partial_y g_{xx} \\
&= \frac{1}{2} y^2 \partial_y y^{-2} = -y^2 y^{-3} = -\frac{1}{y}
\end{aligned}
\tag{9}
$$

and the different components of (8) are

$$
\Gamma^y_{xx} = -\frac{1}{2} g^{yy} \partial_y g_{xx} = -\frac{1}{2} y^2 \partial_y y^{-2} = \frac{1}{y} \tag{10}
$$

$$
\Gamma^y_{yy} = \frac{1}{2} g^{yy} \partial_y g_{yy} = \frac{1}{2} y^2 \partial_y y^{-2} = -\frac{1}{y} \tag{11}
$$

$$
\Gamma^y_{xy} = \frac{1}{2} g^{yy} \left(\partial_x g_{yy} + \partial_y g_{yx} - \partial_y g_{xy} \right) = 0
$$

We calculated all the 6 independent components of Γ^k_{ij}. The non-vanishing Christoffels are

$$
\Gamma^x_{xy} = \Gamma^x_{yx} = \Gamma^y_{yy} = -\frac{1}{y} \qquad \Gamma^y_{xx} = \frac{1}{y} \tag{12}
$$

Christoffels of the Poincare half-plane

The geodesic equation is

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{13}$$

Plug in (12) yields the geodesic equations of the Poincare half-plane

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0 \tag{14}$$

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0 \tag{15}$$

1.3 Direct Calculation of the Geodesic Equation

Let us use the Lagrangian

$$L = \frac{1}{2}g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \tag{16}$$

Substitution of the metric components (4) yields

$$L = \frac{1}{2y^2}\left(\frac{dx}{ds}\right)^2 + \frac{1}{2y^2}\left(\frac{dy}{ds}\right)^2 \tag{17}$$

The Euler Lagrange equations are

$$\frac{d}{ds}\frac{\partial L}{\partial\left(\frac{dx^i}{ds}\right)} - \frac{\partial L}{\partial x^i} = 0 \tag{18}$$

We start with the x coordinate

$$\frac{\partial L}{\partial x} = 0 \tag{19}$$

$$\frac{d}{ds}\frac{\partial L}{\partial\left(\frac{dx}{ds}\right)} = \frac{d}{ds}\left(\frac{1}{y^2}\frac{dx}{ds}\right) = 0 \tag{20}$$

Then the x coordinate E-L equation is

$$\frac{1}{y^2}\frac{d^2x}{ds^2} - 2y^{-3}\frac{dx}{ds}\frac{dy}{ds} = 0 \tag{21}$$

Multiplying by y^2

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0 \tag{22}$$

Now we turn to the y coordinate

$$\frac{\partial L}{\partial y} = -y^{-3}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) \tag{23}$$

$$\frac{d}{ds}\frac{\partial L}{\partial\left(\frac{dy}{ds}\right)} = \frac{d}{ds}\left(\frac{1}{y^2}\frac{dy}{ds}\right) = \frac{1}{y^2}\frac{d^2y}{ds^2} - 2y^{-3}\left(\frac{dy}{ds}\right)^2 \tag{24}$$

Then the y coordinate E-L equation is

$$\frac{1}{y^2}\frac{d^2y}{ds^2} - 2y^{-3}\left(\frac{dy}{ds}\right)^2 + y^{-3}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{25}$$

$\Rightarrow$

$$\frac{1}{y^2}\frac{d^2y}{ds^2} + y^{-3}\left(\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{26}$$

Multiplying by y^2

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{27}$$

We collect results (22),(27)

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0 \tag{28}$$

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{29}$$

The geodesic equations of the Poincare half-plane

Of course we got the same result as in the previous section (14),(15). We read off the Γ^x_{ij} Christoffel symbols from (28) and the Γ^y_{ij} Christoffel symbols from (29), and recover the result of the direct calculation (12).

1.4 Geodesic Curves in Algebraic Form

In order to find the geodesics curves there are regular four steps:
1. Find integrals of motion arising from symmetries.
2. Write explicitly the normalization constraint .
3. Substitute the integrals of motion into the normalization constraint.
4. Either solve the first order ODE's to find the curves in parametric form $x^i(s)$, or divide the velocities and integrate to find the curves in algebraic form.

The **first step** is to find integrals of motion arising from symmetries.

The metric is independent of the x-coordinate, therefore a translation of the x coordinate is a symmetry. The associated *Killing vector field* is

$$\xi^i = (1, 0) \tag{30}$$

and the conserved quantity is

$$p = \xi \cdot u = g_{ij}\xi^i u^j = g_{xj}u^j = g_{xx}u^x = \frac{1}{y^2}\dot{x} \tag{31}$$

where we denote with dot $\dot{x} \equiv \frac{dx}{ds}$. This is the conserved canonical momentum $\frac{\partial L}{\partial\left(\frac{dx}{ds}\right)}$ we found in (20). p is called an integral of motion, it is constant along trajectories that satisfy the equations of motion, i.e., along geodesics.

The **second step** is to write the first integral, namely, the normalization constraint

$$u \cdot u = 1 \tag{32}$$

$$g_{ij}u^i u^j = \frac{1}{y^2}\dot{x}^2 + \frac{1}{y^2}\dot{y}^2 = 1 \tag{33}$$

The **third step** is to substitute the integral of motion (31) into the normalization constraint (33)

$$\dot{x} = py^2 \tag{34}$$

$$\dot{x}^2 = p^2 y^4 \tag{35}$$

$\Rightarrow$

$$\frac{1}{y^2}\left(p^2 y^4 + \dot{y}^2\right) = 1 \tag{36}$$

This is a first order ODE for y

$$\dot{y} = \pm y\sqrt{1 - p^2 y^2} \tag{37}$$

In order to find the algebraic form of the curve, the **fourth step** is to divide the velocities (34)/(37) (for $\dot{y} > 0$)

$$\frac{dx}{dy} = \frac{\frac{dx}{ds}}{\frac{dy}{ds}} = \frac{\dot{x}}{\dot{y}} = \frac{py}{\sqrt{1 - p^2 y^2}} \tag{38}$$

and integrate

$$x = \int \frac{dx}{dy} dy = \int \frac{py}{\sqrt{1 - p^2 y^2}} dy = -\sqrt{\frac{1}{p^2} - y^2} + x_0 \tag{39}$$

Therefore

Half-circle geodesics

$$\boxed{(x - x_0)^2 + y^2 = \frac{1}{p^2}} \tag{40}$$

(40) are half-circles (since $y \geq 0$) centered on the x-axis at $(x_0, 0)$, with radius $\frac{1}{p}$. For $p = 0$ this infinite circle turn into a vertical line. Explicitly, we can plug $p = 0$ into (38) to find $\frac{dx}{dy} = 0$, thus

Vertical line geodesics

$$\boxed{x = x_0} \tag{41}$$

Indeed, this geodesic line of constant x corresponds to motion with no momentum in the x direction $p = 0$.

1.5 Geodesic Curves in Parametric Form

In order to find the parametric form of the curve, the **fourth step** is to solve (37)

$$\int ds = \int \frac{dy}{y\sqrt{1 - p^2 y^2}} \tag{42}$$

Wolfram says that

$$s = -tanh^{-1}\left(\sqrt{1 - p^2 y^2}\right) \tag{43}$$

for simplicity we took the integration constant to be zero.

$$tanh^2(s) = 1 - p^2 y^2 \tag{44}$$

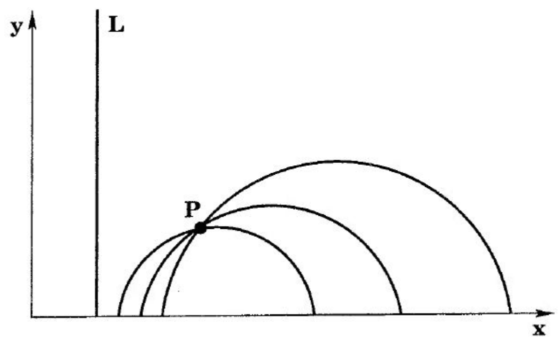
$\Rightarrow$

$$y = \frac{1}{p}\sqrt{1 - tanh^2(s)} \tag{45}$$

$\Rightarrow$

Parametric form of the geodesics

$$\boxed{y(s) = \frac{1}{pcosh(s)}} \tag{46}$$



Remark: This example was important in the history of geometry. Euclid's fifth postulate for Euclidean geometry states that for a straight line L and a point P there is only one straight line (a geodesic) through P that does not intersect L. (That straight line is the one *parallel* to L.) The sphere is an example for which there are no such straight lines through P (all great circles intersect.) The hyperbolic plane is a constant *negative* curvature example (see Chapter 21) where there are an *infinite* number of straight lines through P that do not intersect L (see the example in the figure above).

Figure 1: Geodesics of the Poincare half-plane, from Hartle

Plug into (34)

$$\frac{dx}{ds} = \frac{1}{p cosh^2(s)} \tag{47}$$

$$x(s) = \frac{1}{p} tanh(s) \tag{48}$$

Parametric form of the geodesics

$$\begin{aligned} x^2 + y^2 &= \frac{1}{p^2}\left(tanh^2(s) + \frac{1}{cosh^2(s)}\right) \\ &= \frac{1}{p^2}\left(\frac{sinh^2(s)+1}{cosh^2(s)}\right) = \frac{1}{p^2} \end{aligned} \tag{49}$$

These are the circles centered at the origin.

1.6 Lorentz Hyperboloid

Hyperbolic geometry is a non-Euclidean geometry. The *parallel postulate* of Euclidean geometry is replaced with:

For any given line L and point P not on L, in the plane containing both line L and point P there are **at least two** distinct lines through P that do not intersect L. The hyperbolic plane is a two-dimensional hyperbolic geometry.

There are five main (equivalent) models for this geometry: The Klein disc, the Poincare disc, the Poincare half-plane, the Lorentz hyperboloid, and the pseudo-sphere.

In the last section we explored some aspects of the Poincare half-plane model. Now we shall see how it is related to the *Lorentz hyperboloid* model.

We start in 3-dimensional Minkowski space. In Cartesian coordinates (t, x, y), the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 \tag{50}$$

We transform the spatial coordinates to polar coordinates (t, ρ, ϕ)

$$\begin{aligned} x &= \rho cos\phi \\ y &= \rho sin\phi \end{aligned} \tag{51}$$

In these cylindrical coordinates the metric is

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 \tag{52}$$

Now we look at a **two dimensional surface**, embedded in the 3-dim. Minkowski space, defined by the constraint

$$-t^2 + \rho^2 = -a^2 \tag{53}$$

These are all the points at **equal Minkowski distance** from the origin. The distance squared is negative, so this is a spacelike hyperboloid, inside the light cone. We restrict to the upper half. Let us still treat a as a variable radius, and write the 3-dim. metric in hyperbolic polar coordinates (a, χ, ϕ)

$$t = acosh\chi \tag{54}$$

$$\rho = asinh\chi \tag{55}$$

$$ds^2 = -da^2 + a^2 \left(d\chi^2 + sinh^2\chi d\phi^2\right) \tag{56}$$

This is still a metric of Minkowski space, analogue to spherical polar coordinates in Euclidean 3-dim. space. a is the “radial” coordinate, each constant a corresponds to a hyperbola in the t, ρ plane. $asinh\chi = \rho$ is the distance from

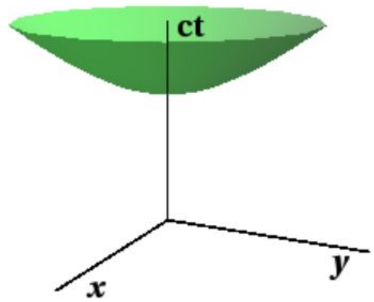


Figure 2: Lorentz Hyperboloid

the t axis. Of course we could have changed directly from Cartesian coordinates to the hyperbolic polar coordinates

$$t = acosh\chi \tag{57}$$

$$x = asinh\chi cos\phi \tag{58}$$

$$y = asinh\chi sin\phi \tag{59}$$

We wanted to emphesize that together with ϕ, the hyperbola in the t, ρ plane becomes a surface of revolution around the t axis. This surface is a 2-dim. Lorentz hyperboloid. The induced metric on it is ($da = 0$)

$$ds^2_{hyperboloid} = a^2 \left(d\chi^2 + sinh^2\chi d\phi^2\right) \tag{60}$$

2-dim. Lorentz hyperboloid metric

The two-dim. metric (60) is analogue to a metric on a sphere with radius a. The difference is that χ is a hyperbolic angle, and there is $sinh\chi$ factor. Compare to a plane of constant t (52) and a sphere (which is embedded in a 3-dim. **Euclidean** space):

$$ds^2_{plane} = d\rho^2 + \rho^2 d\phi^2 \tag{61}$$

$$ds^2_{sphere} = a^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{62}$$

The plane and sphere are also surfaces of revolution around the z axis, with distances from the axis of ρ and $asin\theta$ respectively.

Claim: The hyperboloid is a model of the hyperbolic plane!

1.6.1 Coordinate Transformation

Show that the hyperboloid metric (60) with coordinates (χ, ϕ) is related to the Poincare half-plane metric (1) with coordinates (x, y), by the following transformation

$$x = \frac{cos\phi sinh\chi}{cosh\chi - sinh\chi sin\phi} \tag{63}$$

$$y = \frac{1}{cosh\chi - sinh\chi sin\phi} \tag{64}$$

Notice that these (x, y) coordinates (63),(64) are not the Cartesian coordinates of Minkowski space, but rather the Cartesian-like coordinates of the hyperbolic plane.

Compute the differentials

$$dx = \frac{(cos\phi)\, d\chi + \left(sinh^2\chi - sinh\chi cosh\chi sin\phi\right) d\phi}{(cosh\chi - sinh\chi sin\phi)^2} \tag{65}$$

$$dy = \frac{(-sinh\chi + cosh\chi sin\phi)\, d\chi + (sinh\chi cos\phi)\, d\phi}{(cosh\chi - sinh\chi sin\phi)^2} \tag{66}$$

Plug into $a^2 \frac{dx^2+dy^2}{y^2}$.

2 Kepler Effective Potential

Derive the Kepler effective potential for a test particle in Kepler potential

$$\phi(r) = -\frac{M}{r} \tag{67}$$

The weak field metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right)\left(dr^2 + r^2 d\Omega^2\right) \tag{68}$$

where $d\Omega^2$ is the metric of the unit round sphere

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{69}$$

We will make the Newtonian approximation, i.e., weak field and slow velocity approximation to first order in v^2 or $\frac{M}{r}$. So, we can be smart and approximate

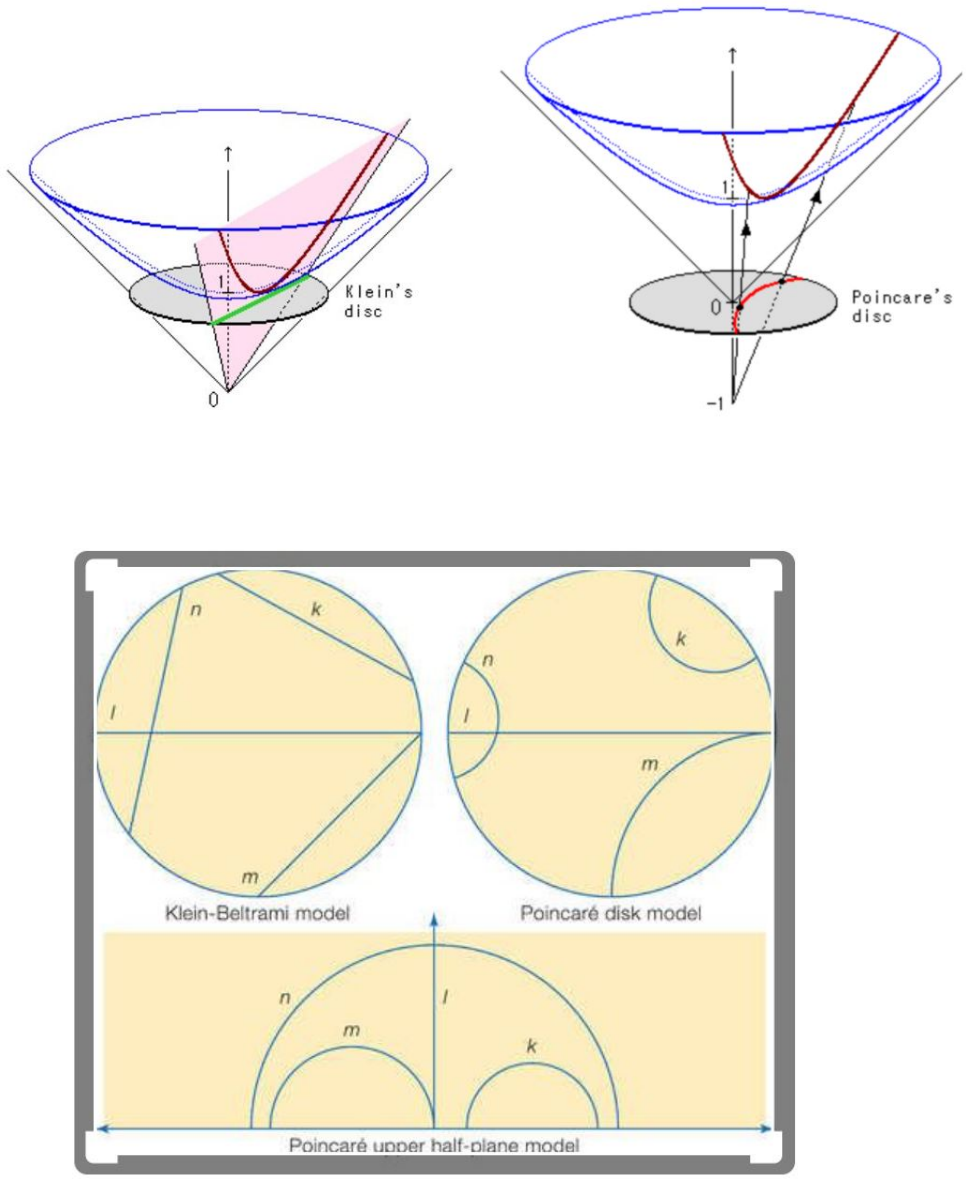


Figure 3: Projections and geodesics of different models of the hyperbolic plane

already the metric to

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + dr^2 + r^2 d\Omega^2 \tag{70}$$

The symmetries of the metric are time translation and all spatial rotations about the origin. These correspond to energy and total angular momentum conservation. Since all components of the angular momentum are conserved, its magnitude and direction are. Therefore the motion is restricted to a plane. We orient the axes such that the plane is $\theta = \frac{\pi}{2}$. The metric reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + dr^2 + r^2 d\phi^2 \tag{71}$$

The coordinates t and ϕ are absent from the metric, thus there are time translation and azimuthal symmetries. The corresponding Killing vector fields are

$$\xi^\mu = (1, 0, 0, 0) \tag{72}$$

$$\lambda^\mu = (0, 0, 0, 1) \tag{73}$$

respectively. The integrals of motion are

$$e = -\xi \cdot u = -g_{\mu\nu}\xi^\mu u^\nu = -g_{tt}u^t = \left(1 - \frac{2M}{r}\right) u^t \tag{74}$$

$$l = \lambda \cdot u = g_{\mu\nu}\lambda^\mu u^\nu = g_{\phi\phi}u^\phi = r^2 u^\phi \tag{75}$$

The normalization constraint is

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{tt}\left(u^t\right)^2 + g_{rr}\left(u^r\right)^2 + g_{\phi\phi}\left(u^\phi\right)^2 \tag{76}$$

In order to plug in the integrals of motion we arrange (74) and (75) as

$$g_{tt}\left(u^t\right)^2 = \frac{e^2}{g_{tt}} = -\frac{e^2}{1 - \frac{2M}{r}} \tag{77}$$

$$g_{\phi\phi}\left(u^\phi\right)^2 = \frac{l^2}{g_{\phi\phi}} = \frac{l^2}{r^2} \tag{78}$$

Plug in (76)

$$-1 = -\frac{e^2}{1 - \frac{2M}{r}} + \dot{r}^2 + \frac{l^2}{r^2} \tag{79}$$

where

$$u^r = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} \approx \frac{dr}{dt} \equiv \dot{r} \tag{80}$$

Since $\frac{dt}{d\tau} \approx 1$ for slow velocities. Also, notice that $v^2 \sim \dot{r}^2 \sim \frac{l^2}{r^2} \sim \frac{M}{r}$ are all small. Multiply (79) by $\left(1 - \frac{2M}{r}\right)$ and drop the second order terms yields

$$-1 + \frac{2M}{r} = -e^2 + \dot{r}^2 + \frac{l^2}{r^2} \tag{81}$$

Rearrange

$$\frac{e^2 - 1}{2} = \frac{1}{2}\dot{r}^2 - \frac{M}{r} + \frac{l^2}{2r^2} \tag{82}$$

$$E = \frac{1}{2}\dot{r}^2 + V_{eff}\left(r\right) \tag{83}$$

where

$$E = \frac{e^2 - 1}{2} \tag{84}$$

and $V_{eff}\left(r\right)$ is the Kepler effective potential

$$V_{eff}\left(r\right) = -\frac{M}{r} + \frac{l^2}{2r^2}$$

Kepler effective potential

The first term is the Kepler potential, and the second term is the potential of the centrifugal force.

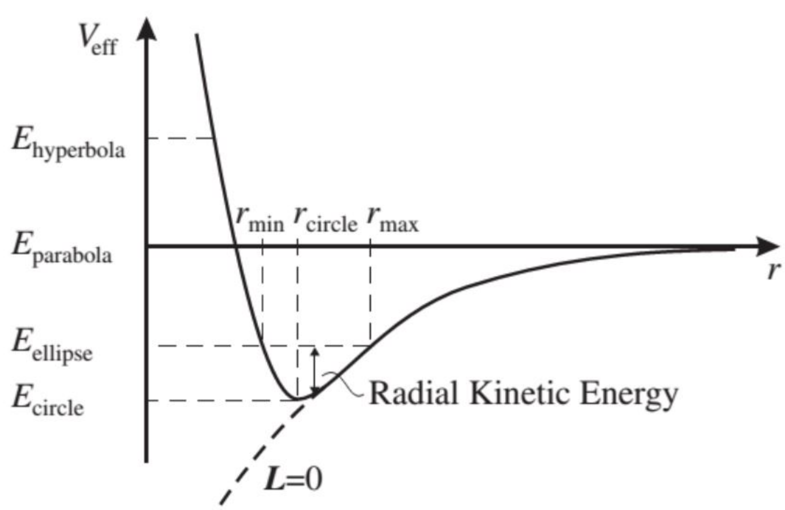


Figure 4: Kepler Effective Potential