כבידה 1

2021/22 סמסטר א'

<u>פתרון תרגיל 2</u>

1. Index Gymnastics

$$X^{\mu}_{\ \nu} = X^{\mu\rho}\eta_{\rho\nu} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

$$X_{\mu}^{\ \nu} = \eta_{\mu\rho}X^{\rho\nu} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

$$X^{\mu}_{\ \mu} = -2 + 0 + 0 + (-2) = -4$$

$$V^{\mu}V_{\mu} = \eta_{\mu\nu}V^{\mu}V^{\nu} = -(-1)^2 + (2)^2 + (0)^2 + (-2)^2 = 7$$

$$V_{\mu}X^{\mu\nu} = \eta_{\rho\sigma}V^{\rho}X^{\sigma\nu} = \begin{pmatrix} 4 \\ -2 \\ 5 \\ 7 \end{pmatrix}$$

$$V_{\nu}X^{\mu\nu} = \eta_{\rho\sigma}V^{\rho}X^{\mu\sigma} = \begin{pmatrix} 4 \\ -5 \\ 1 \\ 4 \end{pmatrix}$$

2. Varying velocity

a) $(gt)^2 < 1 + (gt)^2$ so $dx/dt < 1$.

b)

$$\begin{aligned} u^t &= \frac{1}{\sqrt{1-V^2}} = \sqrt{1+(gt)^2} \\ u^x &= \frac{V}{\sqrt{1-V^2}} = gt \\ u^y &= u^z = 0 \end{aligned}$$

c) The clock of an observer riding on the particle reads proper time. The proper time elapsed from $t = 0$ to t is

$$\tau = \int_0^t dt\,\sqrt{1-V^2} = \int_0^t \frac{dt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\ \sinh^{-1}(gt)\ . \qquad (1)$$

The particle trajectory is

$$x(t) - x_0 = \int_0^t dt\,\frac{gt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\ \sqrt{1+(gt)^2}\ .$$

Thus the relation between τ — the time on the observer's clock — and the location x is

$$\begin{aligned} (x - x_0) &= \frac{1}{g}\ \sqrt{1+\sinh^2(g\tau)} \\ &= \frac{1}{g}\ \cosh(g\tau) \qquad (2) \end{aligned}$$

d) The four force is $f^\alpha = md^2x^\alpha/d\tau^2$ or

$$f^\alpha = (mg\sinh(g\tau), mg\cosh(g\tau), 0, 0)\ .$$

The three force is given by $\vec{F} = md\vec{u}/dt$

$$F^i = (mg, 0, 0)\ .$$

$$f^\alpha = (\gamma F v, \gamma F)$$

$$F = mg$$

Therefore

$$\gamma = cosh\,(g\tau)$$

$$\gamma v = sinh\,(g\tau)$$

$$v = tanh\,(g\tau)$$

Geometric meaning: $\phi = g\tau$ is the hyperbolic angle along the hyperbolic trajectory of the particle in spacetime. It is the instantanuous boost of the particle, relative to an inertial observer. g is the angular frequency.

Kinematic meaning: g is the constant acceleration of the particle.

3. Uniform Acceleration

a.

The four acceleration of accelerated particle is

$$a^{\mu} = \left(\gamma^4 \boldsymbol{v} \cdot \boldsymbol{a}, \gamma^4 \left(\boldsymbol{v} \cdot \boldsymbol{a}\right) \boldsymbol{v} + \gamma^2 \boldsymbol{a}\right) \tag{1}$$

In the accelerated particle's rest frame $\boldsymbol{v} = \boldsymbol{0}$, $a = g$

$$a^{\mu} = (0, g) \tag{2}$$

$$a^{\tau} = 0 \qquad a^{\xi} = g \tag{3}$$

$$u^{\mu} = (1, 0) \tag{4}$$

$$u^{\tau} = 1 \qquad u^{\xi} = 0 \tag{5}$$

b.

We make Lorentz transformation to inertial observer frame, for the 4-vectors u^{μ} and a^{μ}

$$u^t = \gamma u^{\tau} - \beta\gamma u^{\xi} = \gamma \tag{6}$$

$$u^x = \gamma u^{\xi} - \beta\gamma u^{\tau} = -\beta\gamma \tag{7}$$

$$a^t = \gamma a^{\tau} - \beta\gamma a^{\xi} = -\beta\gamma g \tag{8}$$

$$a^x = \gamma a^{\xi} - \beta\gamma a^{\tau} = \gamma g \tag{9}$$

c.

Plug (7) into (8)

$$a^t = \frac{du^t}{d\tau} = g u^x \tag{10}$$

Plug (6) into (9)

$$a^x = \frac{du^x}{d\tau} = g u^t \tag{11}$$

d.

Differentiate (11) and substitute (10)

$$\frac{d^2 u^x}{d\tau^2} = g \frac{du^t}{d\tau} = g^2 u^x \tag{12}$$

Solve with initial condition $u^x(0) = 0$ and normalization $u \cdot u = -1$

$$u^x(\tau) = sinh\left(g\tau\right) \tag{13}$$

$$u^t(\tau) = cosh\left(g\tau\right) \tag{14}$$

Integrate with initial condition $t(0) = 0$, $x(\tau) = x_0$.

$$x(\tau) = x_0 + \frac{1}{g}\left(cosh\left(g\tau\right) - 1\right) \tag{15}$$

$$t(\tau) = \frac{1}{g} sinh\left(g\tau\right) \tag{16}$$

This is a hyperbola of radius x_0. The algebraic form of the curve is

$$t^2 = (x - x_0)^2 + \frac{2}{g}(x - x_0) \tag{17}$$

4. From Hartle's book chapter 5

5-13. *[B,S] One reaction for photoproducing pions is*

$$\gamma + p \rightarrow n + \pi^+ .$$

Find the minimum energy (the threshold energy) a photon would have to have to produce a pion in this way in the frame in which the proton is at rest. Is this energy within reach of contemporary accelerators?

Solution: The threshold condition, Ref. (b) in Box 5.1 just needs to be evaluated in the frame in which the proton is at rest. In that frame

$$p_p^\alpha = (m_p, 0, 0, 0)$$

and the condition gives

$$-2E_\gamma m_p - m_p^2 = -(m_n + m_\pi)^2$$

for the threshold energy E_γ of the photon. Solving for E_γ gives the approximation for $m_n \approx m_p$

$$E_\gamma = m_\pi \left(1 + \frac{m_\pi}{2m_p}\right) \approx 150\text{MeV}.$$

This is well within the reach of contemporary accelerators although photons are not the particles being accelerated.

5. Doppler Effect in Galaxy 1

Galaxy A is approaching since an absorption line with wavelength 500nm for a stationary galaxy is shifted to 450 nm. To nd the speed v at which A is approaching, we use

$$f_{obs} = \sqrt{\frac{1+\beta}{1-\beta}} f_{source}$$

and $\lambda = c/f$ so

$$\lambda_{obs} = \sqrt{\frac{1-\beta}{1+\beta}} \lambda_{source}$$

from which

$$\beta = \frac{\lambda_{source}^2 - \lambda_{obs}^2}{\lambda_{source}^2 + \lambda_{obs}^2}$$

and for $\lambda_{source} = 500nm$, $\lambda_{obs} = 450nm$ you get $\beta \approx 0.105$.

6. **Doppler Effect in Galaxy 2**

א. ענן הגז המתקרב אלינו יהיה בעל תדירות גבוהה יותר ולכן אורך הגל הנמוך יותר (העקומה הכחולה).

ב. נרשום את חוק דופלר עבור התדירויות:

$$\left.\begin{aligned} f_+ &= \sqrt{\frac{1+v}{1-v}}f \\ f_- &= \sqrt{\frac{1-v}{1+v}}f \end{aligned}\right\} \Rightarrow f_- f_+ = f^2 \Rightarrow f = \sqrt{f_- f_+} = \frac{c}{\sqrt{\lambda_- \lambda_+}} \approx 5.99 \cdot 10^{14}\,Hz$$

ג. מכיוון שההפרש בין אורכי הגל הוא קטן, ניתן להשתמש בקירוב הלא יחסותי.

$$\lambda_+ - \lambda_- = v(\lambda_+ + \lambda_-) \Rightarrow v = c\frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-} \approx 540\,{}^{km}\!/_{\sec}$$