

פתרון תרגיל 3

1. מערכת מסתובבת

$$\begin{aligned}
 ds^2 &= -\left[1 - \Omega^2(x^2 + y^2)\right] dt^2 + 2\Omega(ydx - xdy) dt + \\
 &\quad + dx^2 + dy^2 + dz^2 \\
 x &= r \sin\theta \cos\phi \\
 dx &= \sin\theta \cos\phi dr + \cos\theta \cos\phi r d\theta - \sin\theta \sin\phi r d\phi \\
 y &= r \sin\theta \sin\phi \\
 z &= r \cos\theta \\
 dx^2 + dy^2 + dz^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \\
 x^2 + y^2 &= r^2 \sin^2\theta.
 \end{aligned}$$

$$\begin{aligned}
 ydx - xdy &= r \sin\theta \{ \sin\theta \cos\phi \sin\phi dr + \cos\theta \cos\phi \sin\phi r d\theta - \\
 &\quad - \sin\theta \sin^2\phi - \sin\theta \cos\phi \sin\phi dr - \\
 &\quad - r d\theta \cos\theta \cos\phi \sin\phi - \sin\theta \cos^2\phi r d\phi \}
 \end{aligned}$$

The cosines all cancel out and we remain with

$$ydx - xdy = -r^2 d\phi \sin^2\theta.$$

Now we work out the line element:

$$\begin{aligned}
 ds^2 &= -dt^2 + \Omega^2 r^2 \sin^2\theta dt^2 - 2\Omega r^2 \sin^2\theta dt d\phi + \\
 &\quad + r^2 \sin^2\theta d\phi^2 + dr^2 + r^2 d\theta^2 \\
 &= -dt^2 + r^2 \sin^2\theta (\Omega dt - d\phi)^2 + dr^2 + r^2 d\theta^2.
 \end{aligned}$$

We take $\phi \rightarrow \phi' = \phi - \Omega t$ as the question says, and check the line element. Then $d\phi' = d\phi - \Omega dt$, and we get

$$ds^2 = -dt^2 + r^2 \sin^2\theta d\phi'^2 + r^2 d\theta^2 + dr^2$$

(b) To find the geodesic equations we need to minimize the Lagrangian:

$$\begin{aligned}
 L &= \left[(1 - \Omega^2(x^2 + y^2)) \left(\frac{dt}{d\sigma}\right)^2 - 2\Omega \left(y \frac{dx}{d\sigma} - x \frac{dy}{d\sigma}\right) \frac{dt}{d\sigma} - \right. \\
 &\quad \left. - \left(\frac{dx^2}{d\sigma}\right) - \left(\frac{dy^2}{d\sigma}\right) - \left(\frac{dz^2}{d\sigma}\right) \right]^{1/2}
 \end{aligned}$$

The E-L equation for x:

$$\begin{aligned} \frac{d}{d\sigma} \frac{\partial L}{\partial \frac{dx}{d\sigma}} - \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial x} &= -\frac{x\Omega^2}{L} \left(\frac{dt}{d\sigma} \right)^2 + \Omega \frac{1}{L} \frac{dy}{d\sigma} \frac{dt}{d\sigma} \\ \frac{\partial L}{\partial \frac{dx}{d\sigma}} &= -\Omega y \frac{dt}{d\sigma} \frac{1}{L} - \frac{1}{L} \frac{dx}{d\sigma} = -\Omega \frac{dt}{d\tau} - \frac{dx}{d\tau} \end{aligned}$$

Now $Ld\sigma = d\tau$ and so

$$\begin{aligned} \frac{1}{L} \frac{d}{d\sigma} \frac{\partial L}{\partial \frac{dx}{d\sigma}} &= -\frac{d^2x}{d\tau^2} - \frac{d}{d\tau} \left(\Omega y \frac{dt}{d\tau} \right) \\ &= \frac{1}{L} \frac{\partial L}{\partial x} = -x\Omega^2 \left(\frac{dt}{d\tau} \right)^2 + \Omega \frac{dy}{d\tau} \frac{dt}{d\tau} \\ -\frac{d^2x}{d\tau^2} - \Omega y \frac{d^2t}{d\tau^2} &= 2\Omega \frac{dy}{d\tau} \frac{dt}{d\tau} - x\Omega^2 \left(\frac{dt}{d\tau} \right)^2 \end{aligned}$$

This looks more complicated and different than what we did in class, that is because if you look at the metric you will see the components are more complicated: g_{tt} is a function of x and y, and you also have non diagonal components (g_{xt}, g_{yt}). For z it's simple:

$$\frac{d^2z}{d\tau^2} = 0$$

(c) In the non relativistic limit

$$\begin{aligned} \frac{dt}{d\tau} &= 1, \frac{d^2t}{d\tau^2} = 0, \\ \frac{d^2x}{d\tau^2} &= 2\Omega \frac{dy}{dt} - x\Omega^2 \end{aligned}$$

where the first term is the Coriolis force and the second centrifugal.

2. ערפילית כדורית

ו. נחשב את הפוטנציאל ממשוואת פואסון:

$$\frac{1}{r^2} (r^2 \phi')' = 4\pi G \rho \Rightarrow \phi = 4\pi \int \frac{dr}{r^2} \int_0^r dr' r'^2 G \rho$$

$$= \frac{4\pi G \rho}{3} \int \frac{dr}{r^2} r^3 = \frac{2\pi}{3} G \rho r^2$$

ז. נשתמש בנוסחת השדה החלש:

$$\omega = 1 + \phi = 1 + \frac{2\pi G}{3} \rho R^2$$

ח. נחשב את הצפיפות האפקטיבית של השמש:

$$\rho = \frac{M}{\frac{4\pi}{3} R^3} \approx \frac{2 \cdot 10^{30}}{\frac{4\pi}{3} (5 \cdot 10^8)^3} \text{ kg/m}^3 \approx 3.8 \cdot 10^3 \text{ kg/m}^3$$

נציב את הנתון להסחה לאדום:

$$\omega^{-1} = \frac{2\pi G}{3} \frac{\rho R^2}{c^2} \approx \frac{2\pi \cdot 6.67 \cdot 10^{-11}}{3} \cdot \frac{3.8 \cdot 10^3 \cdot (5 \cdot 10^8)^2}{(3 \cdot 10^8)^2} \approx 1.5 \cdot 10^{-6}$$

ט. יש לדרוש שההסחה לאדום תהיה קטנה בהרבה מ-1:

$$\frac{2\pi G}{3} \rho R^2 \ll 1 \Rightarrow R \ll R_c = \sqrt{\frac{3}{2\pi G \rho}}$$

$$\frac{v^2}{r} = \frac{d\phi}{dr} = \frac{4\pi G \rho}{3} r = \frac{2\phi}{r} \Rightarrow v^2 = 2\phi$$

ולכן הזמן העצמי שלו מקיים:

$$d\tau^2 = (1 + 2\phi - v^2) dt^2 = dt^2$$

כלומר, הוא מסונכרן עם צופה במרכז הענן!

- a) The line element, shown in part (a) of the solution to Problem 6, is independent of t' . So the spatial distance $\Delta x'$ between two curves of constant x' remains constant in t' .
- b) The four-velocity in the (t, x) frame can be calculated from the expression for $x(t')$ and $y(t')$ given in Problem 6, together with the line element derived in part (a) of that problem which gives the connection between t' and proper time τ along a curve of constant x' , y' , and z' , viz.,

$$d\tau = \left(1 + \frac{gx'}{c^2}\right) dt'.$$

The resulting four-velocity of a curve at constant x' has components

$$\frac{dt}{d\tau} = \cosh\left(\frac{gt'}{c}\right), \quad \frac{dx}{d\tau} = c \sinh\left(\frac{gt'}{c}\right).$$

(Components in the y - and z -direction vanish). The components of the four-acceleration are

$$a^t = \frac{d^2t}{d\tau^2} = \frac{g}{c} \left(1 + \frac{gx'}{c^2}\right)^{-1} \sinh\left(\frac{gt'}{c}\right),$$

$$a^x = \frac{d^2x}{d\tau^2} = g \left(1 + \frac{gx'}{c^2}\right)^{-1} \cosh\left(\frac{gt'}{c}\right).$$

Thus,

$$a = (\mathbf{a} \cdot \mathbf{a})^{\frac{1}{2}} = \frac{g}{(1 + gx'/c^2)}$$

which decreases with x' .

4. סכרון שעונים

א. הפוטנציאל במקרה זה הוא פשוט $\phi = gh$ ולכן עבור צופה B:

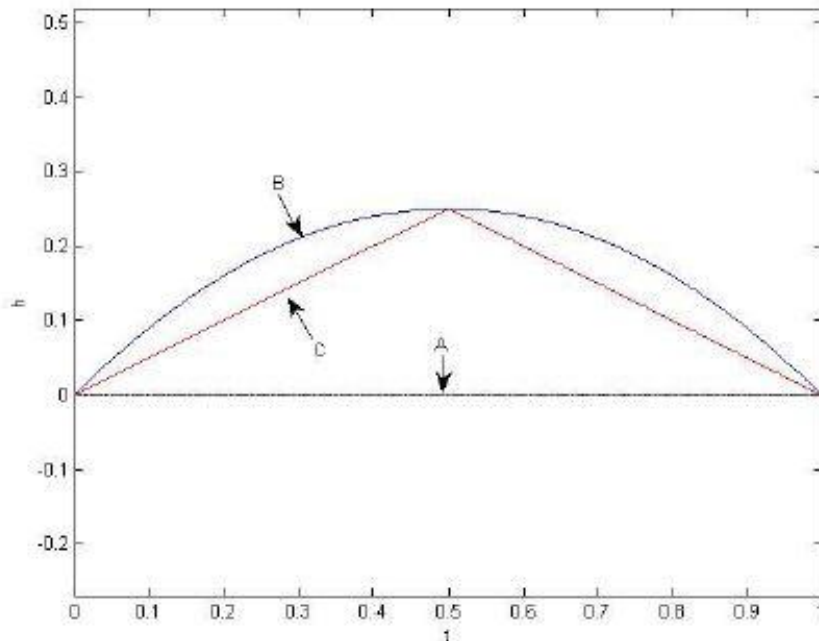
$$\begin{aligned} T_B &= \int_0^T dt(1 + gh_B) = \int_0^T dt \left(1 + g \cdot \frac{1}{2} gt(T-t) \right) \\ &= T + \frac{1}{4} g^2 T^3 - \frac{1}{6} g^2 T^3 = T \left(1 + \frac{1}{12} g^2 T^2 \right) \end{aligned}$$

עבור צופה C:

$$\begin{aligned} T_B &= \int_0^T dt(1 + gh_C) = T + \int_0^{T/2} dtg \frac{gT}{4} t + \int_{T/2}^T dtg \frac{gT}{4} (T-t) \\ &= T + 2 \cdot \frac{1}{8} g^2 T \left(\frac{T}{2} \right)^2 = T \left(1 + \frac{1}{16} g^2 T^2 \right) \end{aligned}$$

שעונו של צופה B מודד את הזמן הארוך ביותר!

ב. אם נשרטט את מסלולי השעונים:



קל לראות שצופה B שווה הכי רחוק ממוקד המשיכה ועל כן לפי הכלל שככל שהשעון קרוב יותר למוקד כך הוא מודד זמן קצר יותר, שעונו של B צריך למדוד זמן ארוך יותר.

ג. במקרה זה יש להכליל את הנוסחה ל- $dt' = \left(1 + \phi - \frac{1}{2} v^2 \right) dt$:

עבור צופה B:

$$\begin{aligned} T_B &= \int_0^T dt \left(1 + gh_B - \frac{1}{2} v_B^2 \right) = \int_0^T dt \left(1 + g \cdot \frac{1}{2} gt(T-t) - \frac{1}{2} g^2 \left(\frac{T}{2} - t \right)^2 \right) \\ &= T \left(1 + \frac{1}{12} g^2 T^2 \right) - \frac{1}{6} g^2 \cdot 2 \cdot \left(\frac{T}{2} \right)^3 = T \left(1 + \frac{1}{24} g^2 T^2 \right) \end{aligned}$$

עבור צופה C:

$$\begin{aligned} T_B &= \int_0^T dt \left(1 + gh_C - \frac{1}{2} v_C^2 \right) \\ &= T + \int_0^{T/2} dt g \frac{gT}{4} t + \int_{T/2}^T dt g \frac{gT}{4} (T-t) - \frac{1}{2} \int_0^T dt \left(\frac{gT}{4} \right)^2 \\ &= T \left(1 + \frac{1}{16} g^2 T^2 \right) - \frac{1}{32} g^2 T^3 = T \left(1 + \frac{1}{32} g^2 T^2 \right) \end{aligned}$$