כבידה 1

2021/22 סמסטר א'

פתרון תרגיל 4

1. **ספירה תלת מימדית**

Solution: To show that the angles (χ, θ, ϕ) are coordinates on the three-sphere, substitute the expressions for W, X, Y, Z into

$$X^2 + Y^2 + Z^2 + W^2 = R^2 .$$

It will be satisfied identically. To find the metric on the three-sphere, work out

$$dX = R\left[(\sin\chi\sin\theta\cos\phi)d\phi + (\sin\chi\cos\theta\sin\phi)d\theta + (\cos\chi\sin\theta\sin\phi)d\chi\right]$$

etc. and substitute into

$$dS^2 = dX^2 + dY^2 + dZ^2 + dW^2 .$$

The result is

$$dS^2 = R^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] .$$

2. **מישור היפרבולי**

a) The distance along an x-constant line to a point on the x-axis is $\int_0 dy/y$ which diverges. Similarly, the distance along any other curve to a point on

b) The non-vanishing Christoffel symbols are:

$$-\Gamma^x_{xy} = \Gamma^y_{xx} = -\Gamma^y_{yy} = 1/y .$$

The geodesic equations are:

$$\frac{d^2x}{dS^2} = \frac{2}{y}\frac{dx}{dS}\frac{dy}{dS}$$
$$\frac{d^2y}{dS^2} = -\frac{1}{y}\left(\frac{dx}{dS}\right)^2 + \frac{1}{y}\left(\frac{dy}{dS}\right)^2 .$$

c) The first geodesic equation in (b) can be written

$$y^2\frac{d}{dS}\left(\frac{1}{y^2}\frac{dx}{dS}\right) = 0 .$$

This can be integrated to give

$$\frac{dx}{dS} = \frac{y^2}{r} \qquad (1)$$

for any constant r. This is one integral of the geodesic equations. Another is supplied by the normalization condition

$$\frac{1}{y^2}\left[\left(\frac{dx}{dS}\right)^2+\left(\frac{dy}{dS}\right)^2\right]=1\ .$$

Using (1) we have,

$$\frac{dy}{dS}=\pm\left[y^2-\frac{y^4}{r^2}\right]^{\frac{1}{2}}\ . \tag{2}$$

To find the shape of the geodesics we compute (for $dy/dS > 0$)

$$\frac{dx}{dy}=\frac{dx/dS}{dy/dS}=\frac{y}{\sqrt{r^2-y^2}}\ .$$

The integral of this is

$$(x-x_0)^2+y^2=r^2$$

where x_0 is a constant. This is a circle of radius r centered at $(x_0, 0)$ on the x-axis. Vertical lines correspond to the limit in which r becomes infinite.

d) Eq (2) can be solved to yield

$$y(S)=\frac{r}{\cosh(S)}$$

and then using this in (1) gives

$$x(S)=r\tanh(S)\ .$$

You can check that $x^2+y^2=r^2$.

3. הפרעה למרחב מינקובסקי

for $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$:

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\left(\eta^{\alpha\beta} + h^{\alpha\beta}\right)\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right) \\ &= \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right) + \mathcal{O}(h^2)\end{aligned}$$

The geodesic equation thus gives

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -\Gamma^i_{00}\left(\frac{dt}{d\tau}\right)^2 - 2\Gamma^i_{0j}\frac{dx^j}{dt}\frac{dt}{dt} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &= -\Gamma^i_{00} + \mathcal{O}(hv) + \mathcal{O}(v^2) = -\Gamma^i_{00} = -\frac{1}{2}\eta^{ii}\left(\partial_i h_{00} + 2\partial_0 h_{0i}\right) \\ &= -\frac{\partial\Phi}{\partial x^i} - \frac{\partial A_i}{\partial t}\end{aligned}$$

to first order in h and v.

4. מטריקת שוורצשילד

$$\Gamma^r_{tt} = \frac{GM}{r^3}(r - 2GM) \qquad \Gamma^r_{rr} = \frac{-GM}{r(r-2GM)} \qquad \Gamma^t_{tr} = \frac{GM}{r(r-2GM)}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -(r-2GM) \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}.$$

5. מטריקה לקינוח

הערה לפתרון: יש טעות, חסר פקטור $\frac{1}{2}$ באיבר הראשון של הלגרנג'יאן. הטעות נגררת לחלק משאר תוצאות הסעיף הראשון.

It is easiest to get the Christoffel symols through the Lagrangian principle, that gives the geodesic equations.

$$L = -\dot{u}\dot{v} + \frac{1}{2}L^2 e^{2\beta}\dot{x}^2 + \frac{1}{2}L^2 e^{-2\beta}\dot{y}^2.$$

(a) The Euler Lagrange eqs. for this Lagrangian are, for coordinates u, x, y and v respectively:

$$\begin{aligned} -v + le^{2\beta}(L' + l\beta')\dot{x}^2 \ddot{+} Le^{-2\beta}(L' - L\beta')\dot{y}^2 &= 0 \\ L^2 e^{2\beta}\ddot{x} + 2Le^{2\beta}(L' + L\beta')\dot{u}\dot{x} &= 0 \\ L^2 e^{-2\beta}\ddot{y} + 2Le^{-2\beta}(L' - L\beta')\dot{u}\dot{y} &= 0 \\ \ddot{u} &= 0 \end{aligned}$$

You have the Christoffel signals:

$$\begin{aligned} \Gamma^v_{xx} &= -Le^{2\beta}(L' + L\beta') \\ \Gamma^v_{yy} &= -Le^{-2\beta}(L' - L\beta') \\ \Gamma^x_{xu} = \Gamma^x_{ux} &= \frac{L'}{L} + \beta' \\ \Gamma^y_{yu} = \Gamma^y_{uy} &= \frac{L'}{L} - \beta' \end{aligned}$$

and the others are zero.

(b) Since $\dot{t} = \dot{u} + \dot{v}$

$$\begin{aligned} \ddot{u} &= 0 \\ \ddot{v} &= 0 \end{aligned}$$

since $\Gamma^u_{\mu\nu} = 0$ for all values of μ and ν and $\Gamma^v_{\mu\nu} \neq 0$ for indices yy and xx. Since $\dot{x} = \dot{y} = 0$ for the given curve, the quadratic term vanishes. So $\dot{u} = \dot{t} = -1$ satisfies the geodesic equations, and it is also timelike $(ds^2/d\tau^2 < 0)$.